# 20 YEARS

OF

# THE FACULTY OF MATHEMATICS AND INFORMATION SCIENCE

## WARSAW UNIVERSITY OF TECHNOLOGY

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A collection of research papers in mathematical analysis and in partial differential equations

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WARSAW 2022

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# Preface

The history of mathematics at the Warsaw University of Technology goes back to 1826 when the Preparatory School for the Polytechnic Institute was founded thanks to the efforts of Stanisław Staszic. Its first director became Kajetan Garbiński, a professor of mathematics. The school was closed in 1831.

The Warsaw Polytechnic Institute named after Tsar Nicolas II was establihed in 1898. Classes were conducted in Russian untill the outbreak of World War I. The Warsaw University of Technology started on its own in 1915. It was the first Polish technical university. All this time at faculties of engineering there were divisions of mathematics which employed famous professors including Georgij Voronoj, Kazimierz Żorawski, Witold Pogorzelski, Stanisław Saks, Antoni Zygmund, Franciszek Leja, Władysław Nikliborc, Stefan Straszewicz and Roman Sikorski.

In 1963 all the divisions of mathematics were joined together in order to establish the Institute of Mathematics, which in 1975 became a part of the Faculty of Technical Physics and Applied Mathematics. In 1999 the institute was transformed into the Faculty of Mathematics and Information Sciences.

The aim of this monograph is to celebrate 20 years of the Faculty of Mathematics and Information Science. We present a collection of research papers in mathematical analysis and in partial differential equations written by mathematicians associated with our faculty. The authors of the papers represent various generations from students to professors.

**Leszek Bartczak**

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# ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO COERCIVE MODELS OF MONOTONE TYPE IN THE THEORY OF INELASTIC DEFORMATIONS. QUASISTATIC CASE

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**Abstract:** We study the asymptotic behaviour of the solution to coercive models of inelastic deformation in quasistatic case. We look for conditions on the boundary data, the external force and nonlinearity such that the solution of our problem converges to a stationary solution.
**Keywords:** viscoplasticity, nonlinear constitutive equations, asymptotic analysis
**Mathematics Subject Classification (2020):** 74D10 (primary), 35M32, 35B40

## 1. INTRODUCTION

The goal of the current paper is to study the inelastic deformation of a solid and its behaviour with specific conditions given for data in coercive quasistatic case. In this article we study the class of models described by H-D. Alber in [1] and completely solved for existence and uniqueness of solutions by H-D. Alber and K. Chełmiński in [2]. We base on those results and extend the results by some asymptotic properties. Very similar problem in dynamical case with dumping added was previously approached by the author (see [3]).

### 1.1. MODEL SETTING

Let the considered body occupy the bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. Let $x \in \Omega$ denote a material point of the body while $t \in (0,+\infty)$ denotes time. Moreover, let denote $\mathfrak{s}(3)$ the set of symmetrical $3\times 3$ matrices with real entries. The main system studied in this paper is in the following form:

$$
\begin{aligned}
&\operatorname{div}_x \mathbb{T}(x,t) = f(x,t),\\
&\mathbb{T}(x,t) = \mathbb{C}\left(\varepsilon(u(x,t)) - \mathrm{B}\, z(x,t)\right),\\
&z_t(x,t) \in g(-\nabla_z \psi(\varepsilon(x,t), z(x,t))).
\end{aligned}
\tag{1}
$$

The first equation in the system (1) represents the balance of forces where the function $\mathbb{T}\colon \Omega\times(0,+\infty)\to\mathfrak{s}(3)$ stands for the stress tensor, while $f\colon \Omega\times(0,+\infty)\to\mathbb{R}^3$ is a given density of volume forces. Since we consider a quasistatic case, we can neglect the inertial term $u_{tt}$. The second equation gives us an elastic constitutive equation. Function $u\colon \Omega\times(0,+\infty)\to\mathbb{R}^3$ is a displacement, while $\varepsilon(u(x,t)) = \frac{1}{2}\left(\nabla_x u(x,t) + \nabla_x^T u(x,t)\right)$ is a linearised Cauchy strain tensor in the case of small deformations. Since $\varepsilon(u)$ is the symmetric part of a displacement's gradient, it takes values in $\mathfrak{s}(3)$. Function $z\colon \Omega\times(0,+\infty)\to\mathbb{R}^N$ describes the inelastic part of the deformation and consists of the plastic part of the strain tensor $\varepsilon^p\colon \Omega\times(0,+\infty)\to\mathfrak{s}(3)$ (notice that $\mathfrak{s}(3)$ is isomorphic with $\mathbb{R}^6$) and other internal parameters $\tilde{z}\colon \Omega\times(0,+\infty)\to\mathbb{R}^{N-6}$, thus we can write $z=(\varepsilon^p,\tilde{z})$. The operator $\mathrm{B}\colon \mathbb{R}^N\to\mathfrak{s}(3)$ is the projection of the $\mathbb{R}^N$ vectors to their first six coordinates *i.e.* $\mathrm{B}(z)=\mathrm{B}(\varepsilon^p,\tilde{z})=\varepsilon^p$. The operator $\mathbb{C}\colon\ \mathfrak{s}(3)\to\mathfrak{s}(3)$ is linear, symmetric and positive definite and is called the elasticity tensor, since the second equation is also called a generalisation of Hooke's law. The differential inclusion given in the system (1) describes an inelastic constitutive equation (flow rule). In the current paper we consider a monotone model, thus we assume that the given function $g\colon D(g)\to 2^{\mathbb{R}^N}$ is a maximal monotone multifunction, where $D(g)\subset\mathbb{R}^N$ denotes the domain of the operator $g$. We assume the free energy function $\psi\colon\ \mathfrak{s}(3)\times\mathbb{R}^N\to\mathbb{R}$ to be given by

$$
\psi(\varepsilon,z) = \frac{1}{2}\mathbb{C}(\varepsilon-\mathrm{B}\,z)\cdot(\varepsilon-\mathrm{B}\,z) + \frac{1}{2}\mathrm{L}\,z\cdot z,
\tag{2}
$$

where L is a positive definite symmetric matrix. It is worth to underline that the condition $\mathrm{L}>0$ is equivalent to $\psi>0$ and in such a case we say that the model is coercive. Additionally, one can observe that the argument of the multifunction $g$ can be expressed as

$$
-\nabla_z\psi(\varepsilon(x,t),z(x,t)) = \mathrm{B}^T\,\mathbb{C}(\varepsilon-\mathrm{B}\,z) - \mathrm{L}\,z = \mathrm{B}^T\,\mathbb{T} - \mathrm{L}\,z.
$$

We complete the considered system with the initial data

$$
z(x,0) = z_0(x), \quad x\in\Omega
\tag{3}
$$

and mixed boundary condition

$$
\begin{aligned}
u(x,t) &= \gamma_{\mathcal{D}}(x,t), \quad (x,t)\in\Gamma_0\times[0,+\infty),\\
\mathbb{T}(x,t)\cdot\mathbf{n}(x) &= \gamma_{\mathcal{N}}(x,t), \quad (x,t)\in\Gamma_1\times[0,+\infty),
\end{aligned}
\tag{4}
$$

where $\Gamma_0,\Gamma_1\subset\partial\Omega$ are relatively open satisfying $\Gamma_0\cap\Gamma_1=\emptyset$, $\Gamma_0\cup\Gamma_1=\partial\Omega$ and $\mathcal{H}^2(\Gamma_0)>0$ (by $\mathcal{H}^2$ we denote two-dimensional boundary measure).

**Remark 1.** *The only required initial condition is indeed the condition for the vector of internal variables $z$. The other conditions (i.e., $\mathbb{T}|_{t=0} = \mathbb{T}_0$ and $u|_{t=0} = u_0$) can be easily evaluated through solving the linear elasticity problem for small deformation in the following form:*

$$\begin{aligned} \operatorname{div}_x \mathbb{T}_0(x) &= f(x,0), \\ \mathbb{T}_0(x) &= \mathbb{C}\left(\varepsilon(u_0(x)) - \mathrm{B}\, z_0(x)\right), \\ u_0(x) &= \gamma_{\mathcal{D}}(x,0), \quad x \in \Gamma_0, \\ \mathbb{T}_0(x)\cdot \mathbf{n}(x) &= \gamma_{\mathcal{N}}(x,0), \quad x \in \Gamma_1. \end{aligned} \tag{5}$$

*Moreover, properties of the operator $\mathbb{C}$ give us that the problem* (5) *is elliptic with respect to $u(x,0)$, hence the following estimate holds*

$$\|T_0\|_2 + \|u_0\|_{1,2} \leqslant C\left(\|f(\cdot,0)\|_2 + \|\mathrm{B}\, z_0\|_2 + \|\gamma_{\mathcal{D}}(\cdot,0)\|_{1/2,2,\Gamma_0} + \|\gamma_{\mathcal{N}}(\cdot,0)\|_{-1/2,2,\Gamma_1}\right). \tag{6}$$

**Remark 2.** *In the inequality* (6) *and further in the paper we use the notation $\|\cdot\|_2$ and $\|\cdot\|_{1,2}$ for standard norms on $L^2(\Omega)$ and $H^1(\Omega)$ respectively, $\|\cdot\|_{1/2,2,\Gamma_0}$ for the norm of Sobolev-Slobodeckij space $H^{1/2}(\Gamma_0)$ while $\|\cdot\|_{-1/2,2,\Gamma_1}$ denotes the norm on $H^{-1/2}(\Gamma_1)$ – a dual space to $H^{1/2}(\Gamma_0)$.*

**Remark 3.** *Properties of the free energy function $\psi$ allow us to define a scalar product and a norm on $L^2(\Omega,\mathfrak{s}(3)\times\mathbb{R}^N)$ as follows*

$$\langle(\varepsilon,z),(\bar\varepsilon,\bar z)\rangle_\psi = \int_\Omega [\mathbb{C}(\varepsilon - \mathrm{B}\, z)]\cdot(\bar\varepsilon - \mathrm{B}\,\bar z) + (\mathrm{L}\, z)\cdot \bar z \,\mathrm{d}x, \tag{7}$$

$$\|(\varepsilon,z)\|_\psi^2 = \int_\Omega [\mathbb{C}(\varepsilon - \mathrm{B}\, z)]\cdot(\varepsilon - \mathrm{B}\, z) + (\mathrm{L}\, z)\cdot z \,\mathrm{d}x. \tag{8}$$

*The norm defined above is equivalent to the standard norm defined on the Lebesgue space $L^2(\Omega,\mathfrak{s}(3)\times\mathbb{R}^N)$ and is called the energetic norm. We will use it in future calculations since we find it much more convenient in the considered model.*

## 1.2. EXISTENCE OF SOLUTION TO THE CONSIDERED MODEL

The following existence theorem is proved in [2].

**Theorem 4.** *Let us suppose that the boundary data $\gamma_{\mathcal{D}}, \gamma_{\mathcal{N}}$ and the external force $f$ possess the regularity*

$$f \in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{9}$$

$$\gamma_{\mathcal{D}} \in W^{3,\infty}(0,T;H^{1/2}(\Gamma_0;\mathbb{R}^3)), \qquad \gamma_{\mathcal{N}} \in W^{2,\infty}(0,T;H^{-1/2}(\Gamma_1;\mathbb{R}^3)), \tag{10}$$

*for all $T > 0$, and that to the initial data $z_0 \in L^2(\Omega;\mathbb{R}^N)$ there is $z^* \in L^2(\Omega;\mathbb{R}^N)$ such that*

$$z^*(x) \in g\left(\mathrm{B}^T\,\mathbb{T}_0(x) - \mathrm{L}\,z_0(x)\right) \quad \textit{almost everywhere in } \Omega,$$

*where $(u_0,\mathbb{T}_0)$ is a weak solution of the problem* (5). *If the considered model is of monotone type with the maximal monotone constitutive function $g\colon D(g)\to 2^{\mathbb{R}^N}$, which satisfies $0\in g(0)$, and with a positive definite matrix* L, *then the system* (1) *with the boundary condition* (4) *possesses a global in time, unique solution*

$$(u,\mathbb{T},z)\in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathfrak{s}(3)\times\mathbb{R}^N)) \textit{ for all } T>0.$$

## 1.3. MAIN RESULTS

Let us state the main assumption for the multifunction $g$:

There exists $\alpha > 0$ such that for any $w,\bar{w},\varepsilon,\bar{\varepsilon},z,\bar{z}$ satisfying $w \in g(-\nabla_z\psi(\varepsilon,z))$ and $\bar{w}\in g(-\nabla_z\psi(\bar{\varepsilon},\bar{z}))$ it holds true that:

$$(w-\bar{w})\cdot[-\nabla_z\psi(\varepsilon,z)+\nabla_z\psi(\bar{\varepsilon},\bar{z})] \geqslant \alpha\,[\psi(\varepsilon-\bar{\varepsilon}),z-\bar{z})] \tag{11}$$

or in other words

$$\begin{aligned}(w-\bar{w})\cdot&[\mathrm{B}^T\,\mathbb{C}(\varepsilon-\mathrm{B}\,z)-\mathrm{L}\,z-\mathrm{B}^T\,\mathbb{C}(\bar{\varepsilon}-\mathrm{B}\,\bar{z})+\mathrm{L}\,\bar{z}]\\ &\geqslant \alpha\left[\frac{1}{2}\mathbb{C}((\varepsilon-\bar{\varepsilon})-\mathrm{B}(z-\bar{z}))\cdot((\varepsilon-\bar{\varepsilon})-\mathrm{B}(z-\bar{z})+\frac{1}{2}\mathrm{L}(z-\bar{z})\cdot(z-\bar{z})\right].\end{aligned} \tag{12}$$

The conditions (11) and (12) are called the strong monotonicity (compare [4]).

Consider the following problem in $\Omega$: for some $f^\infty \in L^2(\Omega,\mathbb{R}^3)$, $\gamma_{\mathcal{D}}^\infty \in H^{1/2}(\Gamma_0;\mathbb{R}^3)$, $\gamma_{\mathcal{N}}^\infty \in (H^{-1/2},\mathbb{R}^3)$ solve

$$\begin{aligned}-\operatorname{div}_x \mathbb{C}(\varepsilon(u^\infty(x))-\mathrm{B}\,z^\infty(x)) = f^\infty(x),&\\ 0\in g\left(-\nabla_z\psi(\varepsilon^\infty(x),z^\infty(x))\right),&\end{aligned} \tag{13}$$

with boundary conditions:

$$\begin{aligned}u^\infty = \gamma_{\mathcal{D}}^\infty \text{ on } \Gamma_0,&\\ \mathbb{C}(\varepsilon(u^\infty)-\mathrm{B}\,z^\infty)\cdot\mathbf{n} = \gamma_{\mathcal{N}}^\infty \text{ on } \Gamma_1.&\end{aligned} \tag{14}$$

**Remark 5.** *In general we need to assume the existence and the uniqueness of solution for the problem (13, 14), however in some special cases they can be proved: From the second equation, using the property* (11) *of g and the definition of the free energy* $\psi$*, we can find* $z^\infty = (\mathrm{B}^T\,\mathbb{C}\,\mathrm{B}+\mathrm{L})^{-1}\,\mathrm{B}^T\,\varepsilon(u^\infty)$*. Inserting this expression into the first equation we obtain a linear equation on* $u^\infty$ *with the right hand side* $f^\infty$*. If we additionally assume, that* $\mathrm{B}^T\,\mathbb{C}\,\mathrm{B}\,\mathrm{L} = \mathrm{L}\,\mathrm{B}^T\,\mathbb{C}\,\mathrm{B}$ *then this equation becomes an elliptic one. This assumption is fulfilled if, for instance,* $z = \varepsilon^p$ *and* $\mathrm{L}\,\varepsilon^p = c\varepsilon^p$*, where* $c \in \mathbb{R}$ *is constant (this is the case of the standard internal coercive approximation, see e.g. the model of Melan-Prager). Considering the boundary conditions of the mixed type we obtain a unique solution to the discussed problem.*

**Theorem 6** (Time independent problem)**.** *Let* $(u,z)$ *be the solution of the problem (1,3,10) according to Theorem 4. Assume that the multifunction g satisfies condition* (11)*. Let the function f satisfy the following conditions*

$$\lim_{t\to\infty} \|f(t) - f^\infty\|_2 = 0,$$

$$\lim_{t\to\infty} \|f_t(t) - f_t^\infty\|_2 = \lim_{t\to\infty} \|f_t(t)\|_2 = 0.$$

*Moreover let us assume that the boundary conditions do not depend on time and satisfy:*

$$\gamma_{\mathcal{D}}(x,t) = \gamma_{\mathcal{D}}^\infty(x),$$

$$\gamma_{\mathcal{N}}(x,t) = \gamma_{\mathcal{N}}^\infty(x).$$

*Then the following convergence holds true:*

$$\lim_{t\to\infty} \|(\varepsilon(u(t) - u^\infty), z(t) - z^\infty)\|_\psi = 0.$$

**Theorem 7** (Time dependent problem)**.** *Let* $(u,z)$ *and* $(\bar{u},\bar{z})$ *be the solutions of the problem (1,3,10) according to Theorem 4 with the same boundary data* (10) *but with different initial conditions* $z_0$ *and* $\bar{z}_0$ *respectively and different volume forces functions* $f$ *and* $\bar{f}$ *respectively. Let the multifunction g satisfy condition* (11)*. If volume forces functions satisfy*

$$\lim_{t\to\infty} \|f(t) - \bar{f}(t)\|_2 = 0,$$

$$\lim_{t\to\infty} \|f_t(t) - \bar{f}_t(t)\|_2 = 0,$$

*then*

$$\lim_{t\to\infty} \|(\varepsilon(t) - \bar{\varepsilon}(t), z(t) - \bar{z}(t))\|_\psi = 0, \tag{15}$$

*where we denote* $\bar{\varepsilon} = \varepsilon(\bar{u})$*.*

**Remark 8.** *Since u and* $u^\infty$ *from Theorem 6 as well as u and* $\bar{u}$ *from Theorem 7 satisfy the same boundary condition, the Korn inequality and properties of the energy norm lead us to*

$$\lim_{t\to\infty} \|u(t) - u^\infty\|_{1,2} = 0,$$

$$\lim_{t\to\infty} \|u(t) - \bar{u}(t)\|_{1,2} = 0.$$

# 2. PROOFS OF MAIN THEOREMS

*Proof of Theorem 6.* Let us denote $\varepsilon^\infty = \varepsilon(u^\infty)$ and $\mathbb{T}_\infty = \mathbb{C}(\varepsilon^\infty - \mathrm{B}z^\infty)$. First, we estimate the energetic norm of the following difference $(\varepsilon - \varepsilon^\infty, z - z^\infty)$. To do it we integrate the energetic norm with respect to time:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|(\varepsilon-\varepsilon^\infty, z-z^\infty)\|_\psi^2 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega [\mathbb{C}((\varepsilon-\varepsilon^\infty)-\mathrm{B}(z-z^\infty))]\cdot((\varepsilon-\varepsilon^\infty)-\mathrm{B}(z-z^\infty))\right. \\
&\quad \left. + (\mathrm{L}(z-z^\infty))\cdot(z-z^\infty)\,\mathrm{d}x\right) \\
&= \int_\Omega [\mathbb{C}((\varepsilon-\varepsilon^\infty)-\mathrm{B}(z-z^\infty))]\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon-\varepsilon^\infty)\right)\mathrm{d}x \\
&\quad - \int_\Omega (z_t - z_t^\infty)\cdot[-\nabla_z\psi(\varepsilon,z)+\nabla_z\psi(\varepsilon^\infty,z^\infty)]\,\mathrm{d}x \\
&= \int_\Omega [\mathbb{C}((\varepsilon-\varepsilon^\infty)-\mathrm{B}(z-z^\infty))]\cdot(\varepsilon(u_t)-\varepsilon(u_t^\infty))\,\mathrm{d}x \\
&\quad - \int_\Omega (z_t - z_t^\infty)\cdot[-\nabla_z\psi(\varepsilon,z)+\nabla_z\psi(\varepsilon^\infty,z^\infty)]\,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts and using (1) and (13) we get

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varepsilon-\varepsilon^\infty, z-z^\infty)\|_\psi^2 &= \int_\Omega (u_t-u_t^\infty)\cdot(f-f^\infty)\,\mathrm{d}x + \int_{\partial\Omega}(u_t-u_t^\infty)\cdot(\mathbb{T}-\mathbb{T}_\infty)n\,\mathrm{d}S \\
&\quad - \int_\Omega (z_t - z_t^\infty)\cdot[-\nabla_z\psi(\varepsilon,z)+\nabla_z\psi(\varepsilon^\infty,z^\infty)]\,\mathrm{d}x.
\end{aligned}
$$

Using the assumption given for $g$ and the special form of boundary conditions we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varepsilon-\varepsilon^\infty, z-z^\infty)\|_\psi^2 \leqslant \int_\Omega (u_t-u_t^\infty)\cdot(f-f^\infty)\,\mathrm{d}x - \alpha\|(\varepsilon-\varepsilon^\infty, z-z^\infty)\|_\psi^2.
$$

We multiply both sides of the inequality above by $e^{2\alpha t}$ and we integrate with respect to $t$ to obtain

$$
\begin{aligned}
\frac{1}{2}e^{2\alpha t}&\|(\varepsilon(t)-\varepsilon^\infty, z(t)-z^\infty)\|_\psi^2 \\
&\leqslant \frac{1}{2}\|(\varepsilon(0)-\varepsilon^\infty, z(0)-z^\infty)\|_\psi^2 + \int_0^t e^{2\alpha s}\int_\Omega (u_t-u_t^\infty)\cdot(f-f^\infty)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
$$

Integration by parts gives us

$$\frac{1}{2}e^{2\alpha t}\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi}^{2} \leqslant \frac{1}{2}\|(\varepsilon(0)-\varepsilon^{\infty},z(0)-z^{\infty})\|_{\psi}^{2}$$

$$-\int_{\Omega}(u(0)-u^{\infty})\cdot(f(0)-f^{\infty})\,\mathrm{d}x + e^{2\alpha t}\int_{\Omega}(u(t)-u^{\infty}))\cdot(f(t)-f^{\infty})\,\mathrm{d}x$$

$$-\int_{0}^{t}e^{2\alpha s}\int_{\Omega}(u-u^{\infty})\cdot[(f_t-f_t^{\infty})+2\alpha(f-f^{\infty})]\,\mathrm{d}x\,\mathrm{d}s. \tag{16}$$

We are going to use Lemma 9 proved in Appendix, hence, using coercivity of the model, the Korn inequality and the assumption that $\alpha > 0$, we estimate

$$\begin{aligned}\int_{\Omega}(u(t)-u^{\infty})\cdot(f(t)-f^{\infty})\,\mathrm{d}x &\leqslant \|u(t)-u^{\infty}\|_2\|f(t)-f^{\infty}\|_2\\ &\leqslant C\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi}\|f(t)-f^{\infty}\|_2\\ &\leqslant \frac{1}{4}\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi}^{2}+C^2\|f(t)-f^{\infty}\|_2^2. \end{aligned}\tag{17}$$

We insert (17) into (16) and subtract the term $\frac{1}{4}e^{2\alpha t}\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi}^{2}$ from both sides:

$$\frac{1}{4}e^{2\alpha t}\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi}^{2}$$

$$\leqslant \frac{1}{2}\|(\varepsilon(0)-\varepsilon^{\infty},z(0)-z^{\infty})\|_{\psi}^{2}-\int_{\Omega}(u(0)-u^{\infty})\cdot(f(0)-f^{\infty})\,\mathrm{d}x$$

$$+\left(Ce^{\alpha t}\|f(t)-f^{\infty}\|_2\right)^2-\int_{0}^{t}e^{2\alpha s}\int_{\Omega}(u-u^{\infty})\cdot[(f_t-f_t^{\infty})+2\alpha(f-f^{\infty})]\,\mathrm{d}x\,\mathrm{d}s.$$

We multiply by 2 and do further estimations

$$\frac{1}{2}e^{2\alpha t}\|(\varepsilon(t)-\varepsilon^{\infty},z(t)-z^{\infty})\|_{\psi})^{2} \leqslant \left(\sqrt{2}Ce^{\alpha t}\|f(t)-f^{\infty}\|_2\right)^2+\frac{1}{2}K^2$$

$$+2\int_{0}^{t}e^{2\alpha s}\|u(s)-u^{\infty}\|_2\left(\|f(s)-f^{\infty}\|_2+2\alpha\|f_t(s)-f_t^{\infty}\|_2\right)\mathrm{d}s,$$

where

$$K=\sqrt{2}\sqrt{\|(\varepsilon(0)-\varepsilon^{\infty},z(0)-z^{\infty})\|_{\psi}^{2}+2\|u(0)-u^{\infty}\|_2\cdot\|f(0)-f^{\infty}\|_2}.$$

We denote $(\varepsilon-\varepsilon^{\infty},z-z^{\infty})=(\varepsilon^{*},z^{*})$ and once again we use coercivity and the Korn inequality to obtain

$$\frac{1}{2}e^{2\alpha t}\|(\varepsilon^{*},z^{*})\|_{\psi})^{2} \leqslant \left(\sqrt{2}Ce^{\alpha t}\|f(t)-f^{\infty}\|_2\right)^2+\frac{1}{2}K^2$$

$$+C_1\int_{0}^{t}e^{2\alpha s}\|(\varepsilon^{*},z^{*})\|_{\psi}\left(\|f(s)-f^{\infty}\|_2+2\alpha\|f_t(s)-f_t^{\infty}\|_2\right)\mathrm{d}s. \tag{18}$$

We apply Lemma 9 to (18)

$$e^{\alpha t}\|(\varepsilon^*,z^*)\|_\psi \leqslant \sqrt{K^2+4C^2\|f(0)-f^\infty\|_2^2}+2C\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\alpha s}\|f(s)-f^\infty\|_2\right)\right|\mathrm{d}s$$

$$+C_1\int_0^t e^{\alpha s}\left(\|f_t(s)-f_t^\infty\|_2+2\alpha\|f(s)-f^\infty\|_2\right)\mathrm{d}s.$$

Since

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\|f(s)-f^\infty\|_2\right| \leqslant \|f_t(s)-f_t^\infty\|_2,$$

we collect like terms to get

$$e^{\alpha t}\|(\varepsilon^*,z^*)\|_\psi \leqslant \sqrt{K^2+4C^2\|f(0)-f^\infty\|_2^2}$$

$$+D\int_0^t e^{\alpha s}\|f_t(s)-f_t^\infty\|_2\,\mathrm{d}s+\alpha D_1\int_0^t e^{\alpha s}\|f(s)-f^\infty\|_2\,\mathrm{d}s.$$

Division by $e^{\alpha t}$ leads to

$$\|(\varepsilon(u(t)-u^\infty),z(t)-z^\infty)\|_\psi \leqslant e^{-\alpha t}\sqrt{K^2+4C^2\|f(0)-f^\infty\|_2^2}+D\int_0^t e^{\alpha(s-t)}\|f_t(s)-f_t^\infty\|_2\,\mathrm{d}s$$

$$+\alpha D_1\int_0^t e^{\alpha(s-t)}\|f(s)-f^\infty\|_2\,\mathrm{d}s.$$

Let us observe that, by assumptions made about the volume force, for any $\delta>0$ there exists $N>0$ such that

$$\|f(t)-f^\infty\|_2 \leqslant \frac{\delta}{4D_1},$$

$$\|f_t(t)-f_t^\infty\|_2=\|f_t(t)\|_2 \leqslant \frac{\alpha\delta}{4D}$$

for $t>N$. Next, we can choose $t_0 \geqslant N$ such that

$$\sqrt{K^2+4C^2\|f(0)-f^\infty\|_2^2}+D\int_0^N e^{\alpha s}\|f_t(s)-f_t^\infty\|_2\,\mathrm{d}s+\alpha D_1\int_0^N e^{\alpha s}\|f(s)-f^\infty\|_2\,\mathrm{d}s$$

$$\leqslant e^{\alpha t_0}\frac{\delta}{2}.$$

Therefore, for $t > t_0$

$$\|(\varepsilon(u(t)-u^\infty), z(t)-z^\infty)\|_\psi$$

$$\begin{aligned}
&\leqslant e^{-\alpha t}\sqrt{K^2+4C^2\|f(0)-f^\infty\|_2^2}\\
&\quad + D\int_0^N e^{\alpha(s-t)}\|f_t(s)-f_t^\infty\|_2\,\mathrm{d}s + \alpha D_1\int_0^N e^{\alpha(s-t)}\|f(s)-f^\infty\|_2\,\mathrm{d}s\\
&\quad + D\int_N^t e^{\alpha(s-t)}\|f_t(s)-f_t^\infty\|_2\,\mathrm{d}s + \alpha D_1\int_N^t e^{\alpha(s-t)}\|f(s)-f^\infty\|_2\,\mathrm{d}s\\
&\leqslant \frac{\delta}{2}+\frac{\alpha\delta}{2}\int_N^t e^{\alpha(s-t)}\,\mathrm{d}s \leqslant \delta.
\end{aligned}$$

□

Since once again we are going to work with difference of solutions the method used to prove Theorem 7 is very similar to the one used above, thus we just present the main points.

*Proof of Theorem 7.* As in the previous proof, we estimate the energetic norm of the difference $(u-\bar{u}, z-\bar{z})$. Hence similarly as in the proof of Theorem 6 we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\varepsilon-\bar{\varepsilon}, z-\bar{z})\|_\psi^2 &= \int_\Omega (u_t-\bar{u}_t)\cdot(f-\bar{f})\,\mathrm{d}x + \int_{\partial\Omega}(u_t-\bar{u}_t)\cdot(\mathbb{T}-\bar{T})n\,\mathrm{d}S\\
&\quad - \int_\Omega (z_t-\bar{z}_t)\cdot[-\nabla_z\psi(\varepsilon,z)+\nabla_z\psi(\bar{\varepsilon},\bar{z})]\,\mathrm{d}x.
\end{aligned}$$

Then, using the same reasoning as in the proof of Theorem 6, we obtain:

$$\frac{1}{2}e^{2\alpha t}\|(\varepsilon(t)-\bar{\varepsilon}(t), z(t)-\bar{z}(t))\|_\psi)^2$$

$$\begin{aligned}
&\leqslant \left(\sqrt{2}Ce^{\alpha t}\|f(t)-\bar{f}(t)\|_2\right)^2+\frac{1}{2}K^2\\
&\quad + 2\int_0^t e^{2\alpha s}\|u(s)-\bar{u}(s)\|_2\left(\|f(s)-\bar{f}(s)\|_2+2\alpha\|f_t(s)-\bar{f}_t(s)\|_2\right)\mathrm{d}s,
\end{aligned}$$

where

$$K=\sqrt{2}\sqrt{\|(\varepsilon(0)-\bar{\varepsilon}(0), z(0)-\bar{z}(0))\|_\psi^2+2\|u(0)-\bar{u}(0)\|_2\cdot\|f(0)-\bar{f}(0)\|_2}.$$

We denote $(\varepsilon-\bar{\varepsilon}, z-\bar{z}) = (\varepsilon^*, z^*)$ and $f-\bar{f}=f^*$ and by Lemma 9 we estimate

$$\begin{aligned}
\|(\varepsilon^*, z^*)\|_\psi &\leqslant e^{-\alpha t}\sqrt{K^2+4C^2\|f^*(0)\|_2^2}\\
&\quad + D\int_0^t e^{\alpha(s-t)}\|f_t^*(s)\|_2\,\mathrm{d}s + \alpha D_1\int_0^t e^{\alpha(s-t)}\|f^*(s)\|_2\,\mathrm{d}s.
\end{aligned}$$

The concluding argument from the proof of Theorem 6 completes the proof. □

# APPENDIX

We prove a Gronwall type lemma, as it was used in Theorem 6.

**Lemma 9.** *Let $m, f, f_t \in L^1(0,T;\mathbb{R})$ be given functions such that $m \geqslant 0$ and let $a \geqslant 0$ be a given constant. Let $\phi : [0,T] \to \mathbb{R}$ be a continuous function and such that*

$$\frac{1}{2}\phi^2(t) \leqslant \frac{1}{2}a^2 + f^2(t) + \int_0^t m(s)\phi(s)\,\mathrm{d}s \qquad \forall t \in [0,T].$$

*Then it holds true that*

$$|\phi| \leqslant \sqrt{a^2 + 2f^2(0)} + \sqrt{2}\int_0^t |f_t(s)|\,\mathrm{d}s + \int_0^t m(s)\,\mathrm{d}s \qquad \forall t \in [0,T].$$

*Proof.* For any $\varepsilon > 0$ we introduce an auxiliary function $\psi_\varepsilon$ by the following formula:

$$\psi_\varepsilon(t) = \frac{1}{2}(a+\varepsilon)^2 + |f(t)|^2 + \int_0^t m(s)\phi(s)\,\mathrm{d}s.$$

It is easy to see that $\psi_\varepsilon(t) \geqslant \frac{1}{2}|\phi(t)|^2$. Differentiation of $\psi_\varepsilon$ with respect to time gives us

$$\psi'_\varepsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(f^2(t)\right) + m(t)\phi(t) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\left(f^2(t)\right) + m(t)\sqrt{2}\sqrt{\psi_\varepsilon(t)}.$$

Notice that $\psi_\varepsilon(t) \geq \frac{1}{2}\varepsilon^2 > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{\psi_\varepsilon(t)}\right) = \frac{\psi'_\varepsilon(t)}{2\sqrt{\psi_\varepsilon(t)}} \leqslant \frac{1}{\sqrt{2}}m(t) + \frac{f(t)f'(t)}{\sqrt{\psi_\varepsilon(t)}}.$$

The above estimation holds true for almost all $t \in (0,T)$. Observe, that the function $\frac{ff'}{\sqrt{\psi_\varepsilon}}$ integrable on $[0,T]$. It follows from the assumption for the function $f$ and the definition of the function $\psi_\varepsilon$. Integration with respect to $t$ leads to

$$\begin{aligned}
\sqrt{\psi_\varepsilon(t)} \leqslant& \sqrt{\psi_\varepsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)\,\mathrm{d}s + \int_0^t \frac{f(s)f'(s)}{\sqrt{\psi_\varepsilon(s)}}\,\mathrm{d}s \\
\leqslant& \sqrt{\psi_\varepsilon(0)} + \frac{1}{\sqrt{2}}\int_0^t m(s)ds + \int_0^t \frac{|f(s)|\cdot|f'(s)|}{\sqrt{\psi_\varepsilon(s)}}\chi_{\{t\in\mathbb{R}_+ :\ f(t)\neq 0\}}(s)\,\mathrm{d}s \\
& \big(\text{ we use } \psi_\varepsilon(t) \geqslant |f(t)|^2\big) \\
\leqslant& \sqrt{\frac{1}{2}(a+\varepsilon)^2 + f^2(0)|} + \frac{1}{\sqrt{2}}\int_0^t m(s)\,\mathrm{d}s + \int_0^t |f'(s)|\,\mathrm{d}s.
\end{aligned}$$

It follows easily that

$$|\phi| \leqslant \sqrt{2}\sqrt{\psi_\varepsilon(t)} \leqslant \sqrt{(a+\varepsilon)^2 + 2f^2(0)} + \int_0^t m(s)\,\mathrm{d}s + \sqrt{2}\int_0^t |f_t(s)|\,\mathrm{d}s$$

for any $t \in [0,T]$ and $\varepsilon > 0$. We pass to the limit with $\varepsilon \to 0^+$, which completes the proof. $\square$

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# WELL-POSEDNESS OF THE NORTON–HOFF PLASTICITY MODEL WITH ISOTROPIC HARDENING IN COSSERAT MEDIA

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**Abstract:** We investigate a local well-posedness of the Norton–Hoff model with isotropic hardening in a Cosserat media. We consider a problem on a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary. The existence proof is based on the Banach fixed-point theorem and a Yosida approximation.
**Keywords:** continuum mechanics, elasto-plastic deformations, Cosserat model
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## 1. INTRODUCTION

In this article we investigate a local well-posedness of the Norton–Hoff model with an isotropic hardening in a Cosserat media. The general Cosserat model was introduced by Cosserat brothers in [13]. K. Chełmiński and P. Neff presented different cases of the model in the introduction to [20]. They introduced infinitesimal elastic and elasto-plastic Cosserat models there. The purely elastic model can be obtained by dividing the macroscopic displacement gradient $\nabla u$ into infinitesimal microrotation and an infinitesimal non-symmetric micropolar stretch tensor $\bar{e} = \nabla u - A$. Then, the complete theory is obtained by a variational principle. The elasto-plastic case is an extension of the purely elastic model. This extension itself is non-dissipative. Its basic idea is dividing the total micropolar stretch into elastic and plastic part and assuming that microrotational effects remain purely elastic.

The Norton-Hoff model is an issue from the theory of elasto-plastic deformations. It has been described in [26]. A mathematical analysis of the model can be found in [3] and [29]. The Norton–Hoff model with isotropic hardening is the Norton–Hoff model with one more scalar function, i.e. the so-called isotropic hardening. The model is well-posed. It was shown in [5]. Said model in different cases has been studied in [11, 12, 15]. Several additional models in the theory of inelastic deformations are listed in [1].

In the paper the elasto-plastic Cosserat model connected with the Norton-Hoff model with isotropic hardening is studied. The main goal of the article is to show a well-posedness of the problem. In the article [4] it is proven that if the Cosserat effect vanishes, the issue approximates Norton-Hoff with isotropic hardening model. The Prandtl-Reuss model and similar issues are investigated in [6]. The Cosserat elasto-plastic model is also studied in [7] and [22]. The paper [21] is devoted to the study of dynamic Cosserat models. See also article [8], where a poroplasticity model with Cosserat effects is investigated. The linear elastic Cosserat model is considered in [16, 17, 23, 24, 25]. In [9, 10] the authors study the Armstrong–Frederick model with Cosserat effects. Some results on thermo-visco-elasticity for Norton-Hoff model with Cosserat effects are to be found in paper [18].

## 2. THE PROBLEM FORMULATION

We shall use the notation specified in Section 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary and let $T > 0$. In order to describe a quasi-static deformation of an inelastic body with microrotations and with isotropic hardening we have to find the displacement vector $u\colon \Omega \times [0,T] \to \mathbb{R}^3$, the microrotation matrix $A\colon \Omega \times [0,T] \to \mathfrak{so}(3)$, the plastic deformation tensor $\varepsilon_p\colon \Omega \times [0,T] \to \mathrm{Sym}(3)$ and the isotropic hardening $y\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\operatorname{div}\sigma = -f, \tag{1a}$$

$$\sigma = 2\mu(\varepsilon - \varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda \operatorname{tr}[\varepsilon] \cdot \mathbb{I}, \tag{1b}$$

$$-l_c \Delta \operatorname{axl}(A) = \mu_c \operatorname{axl}(\operatorname{skew}(\nabla u) - A), \tag{1c}$$

$$\dot{\varepsilon}_p = F(T_E, -\tfrac{\gamma}{\alpha} y),\ \dot{y} = g(T_E, -\tfrac{\gamma}{\alpha} y), \tag{1d}$$

$$T_E = 2\mu(\varepsilon - \varepsilon_p), \tag{1e}$$

$$u|_{\partial\Omega} = u_d,\ A|_{\partial\Omega} = A_d,\ \varepsilon_p(0) = \varepsilon_p^0,\ y(0) = y^0. \tag{1f}$$

Here, $\varepsilon = \operatorname{sym}(\nabla u)$ denotes the infinitesimal elastic strain tensor. The numbers $\lambda,\ \mu$ are the positive Lame constants, $\mu_c$ is the Cosserat couple modulus and $l_c = \mu L_c^2 > 0$ is a material parameter, where $L_c$ with the units of length defines an internal length scale. The constants $\gamma$ and $\alpha$ are positive. The functions $u_d,\ A_d$ are given Dirichlet boundary data and $\varepsilon_p^0$ and $y^0$ are given initial values and function $f$ describes external body forces acting on the material. The functions $F$ and $g$ are given by $F(E,x) = (|\operatorname{dev} E| + \alpha x - k)_+^r \frac{\operatorname{dev} E}{|\operatorname{dev} E|}$ and $g(E,x) = \alpha(|\operatorname{dev} E| + \alpha x - k)_+^r$ for $(E,x) \in \mathrm{Sym}(3) \times \mathbb{R}$, where $r > 1$. We will see in Proposition 4 that $(F,g)$ is a monotone field on $\mathrm{Sym}(3) \times \mathbb{R}$.

Let us see that the initial values of $\varepsilon_p, y$ are explicitly given by (1f), but the initial values of $u$ and of $A$ seem to be unknown. However, let us put $t=0$ to (1a), (1b), (1c) and to (1f)

$$\begin{aligned} \operatorname{div}\sigma(0) &= -f(0), \\ \sigma(0) &= 2\mu(\varepsilon(0)-\varepsilon_p^0)+2\mu_c(\operatorname{skew}(\nabla u(0))-A(0))+\lambda\operatorname{tr}[\varepsilon(0)]\cdot\mathbb{I}, \\ -l_c\Delta\operatorname{axl}(A(0)) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u(0))-A(0)), \\ u(0)=u_d(0),\ A(0) &= A_d(0). \end{aligned} \tag{2}$$

The above equation has a unique solution $u(0)\in H^1(\Omega,\mathbb{R}^3)$ and $A(0)\in H^2(\Omega,\mathfrak{so}(3))$, what follows from the Lax-Milgram Theorem and theorems about regularity of solutions for elliptic equations (see [14]). Moreover, $u(0)$ and $A(0)$ satisfy the following inequality

$$(\|u(0)\|_{H^1(\Omega)}+\|A(0)\|_{H^2(\Omega)})\le C(\|f(0)\|_{H^{-1}(\Omega)}+\|u_d(0)\|_{H^{\frac{1}{2}}(\Omega)}+\|A_d(0)\|_{H^{\frac{3}{2}}(\Omega)})\,, \tag{3}$$

where the constant $C$ depends only on $\Omega$ and the parameters of system (1).

In the paper we want to investigate the existence and the uniqueness of solutions of problem (1). Thus, the main purpose of the article is to prove the following theorem.

**Theorem 1.** *Let us assume that*

$$\begin{aligned} &f\in W^{2,\infty}([0,T],L^2(\Omega,\mathbb{R}^3)), && u_d\in W^{3,\infty}([0,T],\ H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)), \\ &A_d\in W^{3,\infty}([0,T],\ H^{\frac{3}{2}}(\partial\Omega,\mathfrak{so}(3))), && \varepsilon_p^0\in L^2(\Omega,\operatorname{Sym}(3)),\ y^0\in L^2(\Omega), \end{aligned}$$

$F(2\mu(\varepsilon(u(0))-\varepsilon_p^0),-\frac{\gamma}{\alpha}y^0)\in L^2(\Omega,\ \operatorname{Sym}(3))$ *and* $g(2\mu(\varepsilon(u(0))-\varepsilon_p^0),-\frac{\gamma}{\alpha}y^0)\in L^2(\Omega)$, *where* $u(0)$ *and* $A(0)$ *are defined by system (2). Then there exists unique weak solution of (1) such that*

$$\begin{aligned} &u\in W^{1,\infty}([0,T],H^1(\Omega,\mathbb{R}^3)), && A\in W^{1,\infty}([0,T],H^2(\Omega,\mathfrak{so}(3))), \\ &\varepsilon_p\in W^{1,\infty}([0,T],L^2(\Omega,\operatorname{Sym}(3))), && y\in W^{1,\infty}([0,T],\ L^2(\Omega)). \end{aligned}$$

## 3. PRELIMINARIES AND NOTATIONS

In this section we shall recall some basic facts used in this paper and make some remarks about the notation.

We denote by $\mathbb{R}^{3\times3}$ the set of real $3\times3$ matrices. The sets $\operatorname{Sym}(3)$ and $\mathfrak{so}(3)$ are defined as follows:

$$\operatorname{Sym}(3)=\left\{A\in\mathbb{R}^{3\times3}\colon A^T=A\right\}\quad\text{and}\quad\mathfrak{so}(3)=\left\{A\in\mathbb{R}^{3\times3}\colon A^T=-A\right\}.$$

For $A\in\mathbb{R}^{3\times3}$ we define the symmetric part of $A$ as

$$\operatorname{sym}(A)=\frac{1}{2}\left(A+A^T\right)\,,$$

and the skew-symmetric part of a matrix as

$$\operatorname{skew}(A) = \frac{1}{2}\left(A - A^T\right) .$$

It is easy to see that $A = \operatorname{sym}(A) + \operatorname{skew}(A)$, $\operatorname{sym}(A) \in \operatorname{Sym}(3)$ and $\operatorname{skew}(A) \in \mathfrak{so}(3)$. Let $B \in \mathfrak{so}(3)$, then there exist real numbers $a,\ b,\ c$ such that

$$B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} .$$

We define $\operatorname{axl}(B) = (-c,\ b,\ -a)$. Let $A \in \operatorname{Sym}(3)$. We define the deviator of $A$ as

$$\operatorname{dev} A = A - \frac{1}{3}\operatorname{tr}[A]\mathbb{I} ,$$

where $\operatorname{tr}[A]$ is the trace of $A$ and it is defined by $\operatorname{tr}[A] = \sum_{i=1}^{3} A(i,i)$, and $\mathbb{I}$ is the identity matrix. It is easy to see that $\operatorname{dev} A$ is a projection of $A$ onto symmetric matrices with trace equal to zero. Now, let $\Omega \subset \mathbb{R}^3$ be an open set. Let us introduce the space $L^2_{\operatorname{div}}(\Omega)$:

$$L^2_{\operatorname{div}}(\Omega) = \{u \in L^2(\Omega, \mathbb{R}^3) \colon\ \operatorname{div} u \in L^2(\Omega)\} ,$$

where div means the weak divergence. In this space the norm can be defined as follows

$$\|u\|_{L^2_{\operatorname{div}}(\Omega)} = \|u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} .$$

The subsequent fact holds.

**Theorem 2.** *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with the boundary of $C^1$-class. Then, there exists a bounded linear operator $\gamma$: $L^2_{\operatorname{div}}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ such that*

**(i)**

$$\|\gamma u\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{L^2_{\operatorname{div}}(\Omega)} \ \textit{for } u \in L^2_{\operatorname{div}}(\Omega) ,$$

**(ii)**

$$\gamma u = u \cdot n|_{\partial\Omega} \ \textit{for } u \in C(\bar{\Omega}) ,$$

*where $n$ denotes the exterior unit normal vector to $\partial\Omega$.*

*Moreover, if $w \in H^1(\Omega)$ and $w|_{\partial\Omega} = \phi$ (in the sense of traces, see [14]), then for $u \in L^2_{\operatorname{div}}(\Omega)$ the following equality is satisfied:*

$$\langle \gamma u, \phi \rangle = \int_\Omega u \cdot \nabla w dx + \int_\Omega \operatorname{div} u w dx . \tag{4}$$

The condition (ii) from Theorem 2 and (4) justify the notation $\gamma u$ for $u \in L^2_{\operatorname{div}}(\Omega)$ as $u \cdot n$ and $\langle \gamma u,\ \phi \rangle$ for $\phi \in H^{\frac{1}{2}}(\partial\Omega)$ as $\int_{\partial\Omega} u \cdot n\phi\, dS$. The details and the proof of Theorem 2 are given in [28].

In the article we use some basic results of functional analysis. These facts can be found in [19] and [27]. The following lemma is well-known. We include it here with a proof.

**Lemma 3.** *For $v \in H^1_0(\Omega,\mathbb{R}^3)$ the following equality*

$$\|\operatorname{rot}(v)\|^2_{L^2(\Omega)} + \|\operatorname{div}(v)\|^2_{L^2(\Omega)} = \|\nabla v\|^2_{L^2(\Omega)}$$

*holds.*

*Proof.* Let $v \in C^\infty_c(\Omega,\mathbb{R}^3)$. We have

$$\operatorname{rot}(\operatorname{rot}(v)) = -\Delta v + \nabla \operatorname{div}(v). \tag{5}$$

We also know that for an arbitrary $w \in C^\infty_c(\Omega,\mathbb{R}^3)$ the following equality

$$\int_\Omega w \cdot \operatorname{rot}(v) dx = \int_\Omega \operatorname{rot}(w) \cdot v dx$$

holds. Equality (5) gives us

$$\int_\Omega \operatorname{rot}(\operatorname{rot}(v)) \cdot v dx = -\int_\Omega \Delta v \cdot v dx + \int_\Omega \nabla \operatorname{div}(v) \cdot v dx.$$

Integrating by parts we get

$$\int_\Omega |\operatorname{rot}(v)|^2 dx = \int_\Omega |\nabla v|^2 dx - \int_\Omega (\operatorname{div}(v))^2 dx.$$

This proves the lemma for $v \in C^\infty_c(\Omega,\mathbb{R}^3)$. The fact that the space $C^\infty_c(\Omega,\mathbb{R}^3)$ is dense in $H^1_0(\Omega,\mathbb{R}^3)$ completes the proof. □

**Proposition 4.** *A vector field $(F,g) : \operatorname{Sym}(3) \times \mathbb{R} \to \operatorname{Sym}(3) \times \mathbb{R}$ given by*

$$F(E,x) = (|\operatorname{dev} E| + \alpha x - k)^r_+ \frac{\operatorname{dev} E}{|\operatorname{dev} E|}, \qquad g(E,x) = \alpha(|\operatorname{dev} E| + \alpha x - k)^r_+$$

*is a monotone vector field.*

*Proof.* Let $T_1, T_2 \in \operatorname{Sym}(3)$ and $y_1, y_2 \in \mathbb{R}$. Then we have

$$\begin{aligned}
&(F(T_1,y_1) - F(T_2,y_2), g(T_1,y_1) - g(T_2,y_2)) \cdot (T_1 - T_2, y_1 - y_2)\\
&= \left( (\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ \frac{\operatorname{dev} T_1}{|\operatorname{dev} T_1|} - (\operatorname{dev} T_2 + \alpha y_2 - k)^r_+ \frac{\operatorname{dev} T_2}{|\operatorname{dev} T_2|} \right) (T_1 - T_2)\\
&\quad + (\alpha(\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ - \alpha(\operatorname{dev} T_2 + \alpha y_2 - k)^r_+)(y_1 - y_2)\\
&= (\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ \frac{\operatorname{dev} T_1 \cdot T_1}{|\operatorname{dev} T_1|} - (\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ \frac{\operatorname{dev} T_1 \cdot T_2}{|\operatorname{dev} T_1|}\\
&\quad - (\operatorname{dev} T_2 + \alpha y_2 - k)^r_+ \frac{\operatorname{dev} T_2 \cdot T_1}{|\operatorname{dev} T_2|} + (\operatorname{dev} T_2 + \alpha y_2 - k)^r_+ \frac{\operatorname{dev} T_2 \cdot T_2}{|\operatorname{dev} T_2|} +\\
&\quad + (\alpha(\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ - \alpha(\operatorname{dev} T_2 + \alpha y_2 - k)^r_+)(y_1 - y_2)\\
&\geq (\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ |\operatorname{dev} T_1| - (\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ |\operatorname{dev} T_2|\\
&\quad - (\operatorname{dev} T_2 + \alpha y_2 - k)^r_+ |\operatorname{dev} T_1| + (\operatorname{dev} T_2 + \alpha y_2 - k)^r_+ |\operatorname{dev} T_2|\\
&\quad + ((\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ - (\operatorname{dev} T_2 + y_2 - k)^r_+)(\alpha y_1 - \alpha y_2)\\
&= ((|\operatorname{dev} T_1 + \alpha y_1 - k)^r_+ - (|\operatorname{dev} T_2| + \alpha y_2 - k)^r_+)(|\operatorname{dev} T_1| + \alpha y_1 - (|\operatorname{dev} T_2| + \alpha y_2)) \geq 0,
\end{aligned}$$

where the last inequality holds because of monotonicity of $(\cdot - k)^r_+$ on $\mathbb{R}$. □

# 4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

We will prove that problem (1) has a unique weak solution. First, we will introduce a definition of a weak solution of the system.

**Definition 5.** *We say that* $u \in W^{1,\infty}((0,T),H^1(\Omega,\mathbb{R}^3))$, $A \in W^{1,\infty}((0,T),H^2(\Omega,\mathfrak{so}(3,\mathbb{R})))$, $\varepsilon_p \in W^{1,\infty}((0,T),L^2(\Omega,\mathrm{Sym}(3)))$ *and* $y \in W^{1,\infty}((0,T),\ L^2(\Omega))$ *are a weak solution of system (1) if a weak divergence*

$$\mathrm{div}(\sigma) = \mathrm{div}\left(2\mu(\varepsilon-\varepsilon_p)+2\mu_c(\mathrm{skew}(\nabla u)-A)+\lambda\,\mathrm{tr}[\varepsilon]\cdot\mathbb{I}\right) \in L^\infty((0,T),\ L^2(\Omega,\mathbb{R}^3)),$$

*equations in 1 are satisfied almost everywhere in* $\Omega\times[0,T]$ *and the initial-boundary conditions are satisfied in the sense of traces.*

## 4.1. REGULARIZED PROBLEM

A field $(F,g)$ is monotone, so it can be approximated by a Lipschitz function so-called Yosida approximation (see [2]). Let $(F^\eta,g^\eta)$ be a Yosida approximation of $(F,g)$. In (1d) we replace $F$ and $g$ with $F^\eta$ and $g^\eta$. Thus, we get the regularized problem

$$\mathrm{div}\,\sigma^\eta = -f, \tag{6a}$$
$$\sigma^\eta = 2\mu(\varepsilon^\eta-\varepsilon_p^\eta)+2\mu_c(\mathrm{skew}(\nabla u^\eta)-A^\eta)+\lambda\,\mathrm{tr}[\varepsilon^\eta]\cdot\mathbb{I}, \tag{6b}$$
$$-l_c\Delta\,\mathrm{axl}(A^\eta) = \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u^\eta)-A^\eta), \tag{6c}$$
$$\dot{\varepsilon}_p^\eta = F^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta),\ \dot{y}^\eta = g^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta), \tag{6d}$$
$$T_E^\eta = 2\mu(\varepsilon^\eta-\varepsilon_p^\eta), \tag{6e}$$
$$u^\eta|_{\partial\Omega} = u_d,\ A^\eta|_{\partial\Omega} = A_d,\ \varepsilon_p^\eta(0) = \varepsilon_p^0,\ y^\eta(0) = y^0. \tag{6f}$$

First, we will show the well-posedness of the above problem and then we will pass to the limit as $\eta\to 0^+$.

**Theorem 6.** *Let us assume that*

$$f\in C([0,T],L^2(\Omega,\mathbb{R}^3)),\qquad u_d\in C([0,T],H^{\frac12}(\partial\Omega,\mathbb{R}^3)),$$
$$A_d\in C([0,T],H^{\frac32}(\partial\Omega,\mathfrak{so}(3,\mathbb{R}))),\qquad \varepsilon_p^0\in L^2(\Omega,\mathrm{Sym}(3)),\ y^0\in L^2(\Omega),$$

*then problem (6) has a unique solution*

$$u^\eta\in C([0,T],H^1(\Omega,\mathbb{R}^3)),\qquad A^\eta\in C([0,T],H^2(\Omega,\mathfrak{so}(3,\mathbb{R}))),$$
$$\varepsilon_p^\eta\in C^1([0,T],L^2(\Omega,\mathrm{Sym}(3))),\qquad y^\eta\in C^1([0,\ T],L^2(\Omega)).$$

*Moreover, if we assume in addition that*

$$
\begin{aligned}
f &\in C^1([0,T],L^2(\Omega,\mathbb{R}^3)), \qquad u_d \in C^1([0,T],H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)),\\
A_d &\in C^1([0,\ T],\ H^{\frac{3}{2}}(\partial\Omega,\mathfrak{so}(3,\mathbb{R})),
\end{aligned}
\tag{7}
$$

*then* $u^\eta \in C^1([0,T],H^1(\Omega,\mathbb{R}^3))$ *and* $A^\eta \in C^1([0,\ T],\ H^2(\Omega,\mathfrak{so}(3,\mathbb{R}))$.

*Proof.* For simplicity we omit $\eta$ in the proof. Let $\mathbb{X} = C([0,T],L^2(\Omega,\mathrm{Sym}(3)))$. We define a mapping $P\colon \mathbb{X} \to \mathbb{X}$. Let $\varepsilon \in \mathbb{X}$. We solve the system of ordinary differential equations

$$
\dot{\varepsilon}_p = F^\eta(2\mu(\varepsilon-\varepsilon_p), -\tfrac{\gamma}{\alpha}y), \qquad \varepsilon_p(0) = \varepsilon_p^0, \tag{8}
$$

$$
\dot{y} = g^\eta(2\mu(\varepsilon-\varepsilon_p), -\tfrac{\gamma}{\alpha}y), \qquad y(0) = y^0. \tag{9}
$$

We get $\varepsilon_p$ and $y$ from this system. Then, we solve the following system with these functions

$$
\begin{aligned}
\operatorname{div}\sigma &= -f,\\
\sigma &= 2\mu(\varepsilon(u)-\varepsilon_p) + 2\mu_c(\operatorname{skew}(\nabla u) - A) + \lambda\operatorname{tr}[\varepsilon(u)]\cdot\mathbb{I},\\
-l_c\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u) - A),\\
u|_{\partial\Omega} &= u_d,\ A|_{\partial\Omega} = A_d.
\end{aligned}
\tag{10}
$$

The existence of $A$ and $u$ follows from the Lax–Milgram theorem. Finally, we define $P(\varepsilon)$ as $\varepsilon(u)$.

Now, we show that $P$ is a contraction for sufficiently small $T > 0$. Let $\varepsilon^1, \varepsilon^2 \in \mathbb{X}$. Let $\varepsilon_p^1, y^1, u^1, A^1$ be functions which we get defining $P(\varepsilon^1)$ and let $\varepsilon_p^2, y^2, u^2, A^2$ be analogous functions for $\varepsilon^2$. System (10) yields

$$
\begin{aligned}
\int_\Omega 2\mu(\varepsilon(u^1-u^2) - (\varepsilon_p^1-\varepsilon_p^2))\cdot\varepsilon(u^1-u^2) + 2\mu_c|\operatorname{skew}(\nabla u^1-\nabla u^2) - (A^1-A^2)|^2&\\
+ l_c|\nabla A^1 - \nabla A^2|^2 + \lambda\overset{2}{\operatorname{div}}(u^1-u^2)dx = 0.&
\end{aligned}
$$

Therefore, we have

$$
\int_\Omega |\varepsilon(u^1-u^2)|^2 dx \le \int_\Omega \varepsilon(u^1-u^2)\cdot(\varepsilon_p^1-\varepsilon_p^2)dx. \tag{11}
$$

Otherwise, equations (9) result us

$$
\begin{aligned}
&\|y^1(t)-y^2(t)\|_{L^2(\Omega)} \le \int_0^t \|g^\eta(2\mu(\varepsilon^1-\varepsilon_p^1), -\tfrac{\gamma}{\alpha}y^1) - g^\eta(2\mu(\varepsilon^2-\varepsilon_p^2), -\tfrac{\gamma}{\alpha}y^2)\|_{L^2(\Omega)}d\tau\\
&\le CT(\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))} + \|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))} + \|y^1-y^2\|_{C([0,T],L^2(\Omega))})
\end{aligned}
$$

for $0 \le t \le T$. Putting $T < \frac{1}{C}$ we get

$$
\|y^1-y^2\|_{C([0,T],L^2(\Omega))} \le \frac{CT}{1-CT}(\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,Sym(3)))} + \|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,Sym(3)))}). \tag{12}
$$

On the other hand, equation (8) yields

$$
\begin{aligned}
&\|\varepsilon_p^1(t)-\varepsilon_p^2(t)\|_{L^2(\Omega,\mathrm{Sym}(3))}\\
&\quad\le \int_0^t \|F^\eta(2\mu(\varepsilon^1-\varepsilon_p^1),-\tfrac{\gamma}{\alpha}y^1)-F^\eta(2\mu(\varepsilon^2-\varepsilon_p^2),-\tfrac{\gamma}{\alpha}y^2)\|_{L^2(\Omega,\mathrm{Sym}(3))}d\tau\\
&\quad\le CT(\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}+\|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}\\
&\qquad+\|y^1-y^2\|_{C([0,T],L^2(\Omega))})\\
&\quad\le \frac{CT}{1-CT}(\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}+\|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}).
\end{aligned}
\tag{13}
$$

We used here inequality (12). Inequality (13) gives us

$$
\|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}\le \frac{CT}{1-2CT}\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))},
\tag{14}
$$

when we take $T<\frac{1}{C2}$. Now, inequality (11) yields

$$
\|\varepsilon(u^1)-\varepsilon(u^2)\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))}\le \|\varepsilon_p^1-\varepsilon_p^2\|_{C([0,T],L^2(\Omega,\mathrm{Sym}(3)))},
$$

which with (14) results in

$$
\|\varepsilon(u^1)-\varepsilon(u^2)\|_{C([0,T],L^2(\Omega,Sym(3)))}\le \frac{CT}{1-2CT}\|\varepsilon^1-\varepsilon^2\|_{C([0,T],L^2(\Omega,Sym(3)))}.
$$

Finally, we see that $P$ is Lipshitz function. If we take $T<\frac{1}{3C}$, it will be a contraction.

From the Banach fixed-point theorem we get that (6) possesses a unique solution on $[0,T]$. Because $T>0$ does not depend on initial values, we can extend the solution on $[0,T]$ for an arbitrary $T$.

We have to show the last part of the theorem. Let us assume (7) and let $\hat{u}$ and $\hat{A}$ be a solution of the system

$$
\begin{aligned}
\operatorname{div}\hat{\sigma}&=-\dot{f},\\
\hat{\sigma}&=2\mu(\varepsilon(\hat{u})-\dot{\varepsilon}_p)+2\mu_c(\operatorname{skew}(\nabla\hat{u})-\hat{A})+\lambda\,\mathrm{tr}[\varepsilon(\hat{u})]\cdot\mathbb{I},\\
-l_c\Delta\,\mathrm{axl}(\hat{A})&=\mu_c\,\mathrm{axl}(\operatorname{skew}(\nabla\hat{u})-\hat{A}),\\
\hat{u}|_{\partial\Omega}&=\dot{u}_d,\ \hat{A}|_{\partial\Omega}=\dot{A}_d.
\end{aligned}
\tag{15}
$$

Now, let us take $0\le t\le T$ and sufficiently small $h\in\mathbb{R}$. System (15) is analogous to system (2), so we have inequality similar to (3)

$$
\begin{aligned}
&\left\|\frac{u(t+h)-u(t)}{h}-\hat{u}(t)\right\|_{H^1(\Omega,\mathbb{R}3)}+\left\|\frac{A(t+h)-A(t)}{h}-\hat{A}(t)\right\|_{H^2(\Omega,\mathfrak{so}(3))}\\
&\quad\le C\Bigg(\left\|\frac{f(t+h)-f(t)}{h}-\dot{f}(t)\right\|_{L^2(\Omega,\mathbb{R}^3)}+\left\|\frac{\varepsilon_p(t+h)-\varepsilon_p(t)}{h}-\dot{\varepsilon}_p(t)\right\|_{L^2(\Omega,\mathrm{Sym}(3))}\\
&\qquad+\left\|\frac{u_d(t+h)-u_d(t)}{h}-\dot{u}_d(t)\right\|_{H^{\frac12}(\partial\Omega,\mathbb{R}^3)}+\left\|\frac{A_d(t+h)-A_d(t)}{h}-\dot{A}_d(t)\right\|_{H^{\frac32}(\partial\Omega,\mathfrak{so}(3))}\Bigg).
\end{aligned}
$$

Then, we pass to the limit with $h\to 0$. It finishes the proof. □

## 4.2. ESTIMATION OF ENERGY

The energy of system (1) is a very important tool in studying this problem. We define it as follows

$$\begin{aligned}&\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)(t)\\&\quad=\int_\Omega \mu|\varepsilon-\varepsilon_p|^2+\frac{1}{2}\lambda\,\mathrm{tr}[\varepsilon]^2+\mu_c|\,\mathrm{skew}(\nabla u)-A|^2+2l_c|\nabla\,\mathrm{axl}(A)|^2+\frac{1}{2}\frac{\gamma}{\alpha}y^2dx.\end{aligned}$$

**Theorem 7.** *There exists a constant $C>0$ such that for all $u \in H_0^1(\Omega,\mathbb{R}^3)$, $y\in L^2(\Omega)$, $\varepsilon_p\in L^2(\Omega,\mathrm{Sym}(3))$, $A\in H_0^1(\Omega,\ \mathfrak{so}(3))$ the inequality*

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)\geq C(\|u\|^2_{H^1(\Omega)}+\|A\|^2_{H^1(\Omega)}) \tag{16}$$

*holds. Moreover, if $u\in\{v\in H^1(\Omega,\mathbb{R}^3):v|_{\partial\Omega}=u_d\}$, where $u_d\in H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)$, and $A\in\{B\in H^1(\Omega,\mathfrak{so}(3)):B|_{\partial\Omega}=A_d\}$, where $A_d\in H^{\frac{1}{2}}(\Omega,\mathfrak{so}(3))$, then there exists $C_1>0$ such that the inequality*

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)+C_1\geq C(\|u\|^2_{H^1(\Omega)}+\|A\|^2_{H^1(\Omega)})$$

*is satisfied.*

*Proof.* Let $u\in H_0^1(\Omega,\mathbb{R}^3)$, $A\in H_0^1(\Omega,\ \mathfrak{so}(3))$, $\varepsilon_p\in L^2(\Omega,\mathrm{Sym}(3))$, $y\in L^2(\Omega)$, then we have

$$\begin{aligned}\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)(t)&\geq\int_\Omega\frac{1}{2}\lambda\,\mathrm{tr}[\varepsilon]^2+\mu_c|\,\mathrm{skew}(\nabla u)-A|^2+2l_c|\nabla\,\mathrm{axl}(A)|^2dx\\&=\int_\Omega\frac{1}{2}\lambda\,(\mathrm{div}\,u)^2+\mu_c|\,\mathrm{rot}\,u|^2-2\mu_c\,\mathrm{skew}(\nabla u)\cdot A+\mu_c|A|^2+l_c|\nabla A|^2dx\\&\geq\int_\Omega\frac{1}{2}\lambda\,(\mathrm{div}\,u)^2+\frac{1}{2}\mu_c|\,\mathrm{rot}\,u|^2-\mu_c|A|^2+l_c|\nabla A|^2dx.\end{aligned}$$

Now, we use the Poincaré inequality and Lemma 3

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)(t)\geq\min\left(\tfrac{1}{2}\lambda,\tfrac{1}{2}\mu_c\right)\|\nabla u\|^2_{L^2(\Omega)}-\mu_cC_\Omega\|\nabla A\|^2_{L^2(\Omega)}+l_c\|\nabla A\|^2_{L^2(\Omega)}.$$

It results in

$$(1+\frac{\mu_cC_\Omega}{2l_c})\mathcal{E}(u,\varepsilon,\varepsilon_p,A,y)(t)\geq\min\left(\tfrac{1}{2}\lambda,\tfrac{1}{2}\mu_c,l_c\right)(\|\nabla u\|^2_{L^2(\Omega)}+\|\nabla A\|^2_{L^2(\Omega)}).$$

This finishes the proof of inequality (16).

Let $v\in H^1(\Omega,\mathbb{R}^3)$ and $B\in H^1(\Omega,\mathfrak{so}(3))$ be such that $v|_{\partial\Omega}=u_d\}$ and $B|_{\partial\Omega}=A_d$. Then, from the proven part of the theorem we have

$$\mathcal{E}(u-v,\varepsilon,\varepsilon_p,A-B,y)(t)\geq C(\|u-v\|^2_{H^1(\Omega)}+\|A-B\|^2_{H^1(\Omega)}).$$

This immediately completes the proof. □

We estimate the energy of solutions of (6) in the next few theorems.

**Theorem 8.** *Let us assume that* $(u^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)$ *is a solution of (6) for* $\eta > 0$ *and let us assume that conditions (7) are satisfied. Then, there exists a constant C such that for all* $\eta > 0$ *the inequality*

$$\mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)(t) \le C \textit{ for all } 0 \le t \le T$$

*is satisfied.*

*Proof.* Let us calculate the derivative of $\mathcal{E}$

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)(t) \\
&= \int_\Omega 2\mu(\varepsilon^\eta - \varepsilon_p^\eta)(\dot\varepsilon^\eta - \dot\varepsilon_p^\eta) + \lambda \operatorname{tr}[\varepsilon^\eta]\operatorname{tr}[\dot\varepsilon^\eta] + 2\mu_c(\operatorname{skew}(\nabla u^\eta) - A^\eta)(\operatorname{skew}(\nabla \dot u^\eta) - \dot A^\eta) \\
&\quad + 4l_c \nabla \operatorname{axl}(A^\eta) \cdot \nabla \operatorname{axl}(\dot A^\eta) + \frac{\gamma}{\alpha} y^\eta \cdot \dot y^\eta \, dx \\
&= \int_\Omega \big(2\mu(\varepsilon^\eta - \varepsilon_p^\eta) + \lambda \operatorname{tr}[\varepsilon^\eta]\mathbb{I} + 2\mu_c(\operatorname{skew}(\nabla u^\eta) - A^\eta)\big) \cdot \nabla \dot u^\eta \, dx \\
&\quad - \int_\Omega 4\mu_c(\operatorname{axl}(\operatorname{skew}(\nabla u^\eta) - A^\eta)) \cdot \operatorname{axl}(\dot A^\eta) dx - \int_\Omega 4l_c \Delta \operatorname{axl}(A^\eta) \cdot \operatorname{axl}(\dot A^\eta) dx \\
&\quad - \int_\Omega \dot\varepsilon_p^\eta \cdot T_E^\eta \, dx + \int_\Omega \frac{\gamma}{\alpha} y^\eta \cdot \dot y^\eta \, dx + 4l_c \int_{\partial\Omega} (\nabla \operatorname{axl}(A^\eta) \cdot n) \cdot \operatorname{axl}(\dot A^\eta) dS \\
&= I_1 - I_2 - I_3 - I_4 + I_5 + I_6,
\end{aligned}$$

where we have integrated by parts. By equation (6b) we know that

$$\begin{aligned}
I_1 &= \int_\Omega \sigma^\eta \cdot \nabla \dot u^\eta \, dx = \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \dot u^\eta \, dS - \int \operatorname{div} \sigma^\eta \cdot \dot u^\eta \, dx \\
&\overset{(6a)}{=} \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \dot u^\eta \, dS + \int_\Omega f \cdot \dot u^\eta \, dx.
\end{aligned}$$

From (6c) we know that $-I_2 - I_3 = 0$. Next, integrals $-I_4 + I_5 \le 0$ because $(F^\eta, g^\eta)$ is a monotone vector field. Finally, all these inequalities yield

$$\begin{aligned}
\dot{\mathcal{E}}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)(t) &\le \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \dot u^\eta \, dS \\
&\quad + 4l_c \int_{\partial\Omega} \operatorname{axl}(\dot A^\eta) \cdot (\nabla \operatorname{axl}(A^\eta) \cdot n) dS + \int_\Omega f \cdot \dot u^\eta \, dx.
\end{aligned}$$

We integrate the above inequality over $[0,t]$

$$\begin{aligned}
&\mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)(t) \\
&\quad \le \mathcal{E}(u^\eta, \varepsilon^\eta, \varepsilon_p^\eta, A^\eta, y^\eta)(0) + \int_0^t \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \dot u^\eta \, dS \, d\tau \\
&\qquad + 4l_c \int_0^t \int_{\partial\Omega} \operatorname{axl}(\dot A^\eta) \cdot (\nabla \operatorname{axl}(A^\eta) \cdot n) dS \, d\tau + \int_0^t \int_\Omega f \cdot \dot u^\eta \, dx \, d\tau \\
&\quad = \mathcal{E}(0) + J_1 + J_2 + J_3.
\end{aligned} \tag{17}$$

By Theorem 2 we obtain

$$
\begin{aligned}
J_1 &\leq \int_0^t C(\|\sigma^\eta\|_{L^2(\Omega)} + \|\operatorname{div}(\sigma^\eta)\|_{L^2(\Omega)})\|\dot{u}_d\|_{H^{\frac{1}{2}}(\partial\Omega)} d\tau \\
&\leq \int_0^t C\|f\|_{L^2(\Omega)}\|\dot{u}_d\|_{H^{\frac{1}{2}}(\partial\Omega)} d\tau + \int_0^t C\sqrt{\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta,y^\eta)}\|\dot{u}_d\|_{H^{\frac{1}{2}}(\partial\Omega)} d\tau \\
&\leq C(t) + C\int_0^t \sqrt{\mathcal{E}} d\tau \leq C(t) + \int_0^t \mathcal{E} d\tau,
\end{aligned} \tag{18}
$$

where in the last inequality we use the Young inequality. The function $C(t)$ is positive and independent of $\eta$.

Theorem 2 and equation (6c) give us

$$
\begin{aligned}
J_2 &\leq C4l_c \int_0^t (\|\nabla \operatorname{axl}(A^\eta)\|_{L^2(\Omega)} + \|\Delta \operatorname{axl}(A^\eta)\|_{L^2(\Omega)})\|\dot{A}_d\|_{H^{\frac{1}{2}}} d\tau \\
&= C\int_0^t (4l_c\|\nabla \operatorname{axl}(A^\eta)\|_{L^2(\Omega)} + 4\|\mu_c \operatorname{axl}(\operatorname{skew}(\nabla u^\eta) - A^\eta)\|_{L^2(\Omega)})\|\dot{A}_d\|_{H^{\frac{1}{2}}} d\tau \\
&\leq C\int_0^t \sqrt{\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta,y^\eta)}\|\dot{A}_d\|_{H^{\frac{1}{2}}(\partial\Omega)} d\tau \leq C(t) + \int_0^t \mathcal{E} d\tau.
\end{aligned} \tag{19}
$$

Integrating partially by time and using Theorem 7 we obtain

$$
\begin{aligned}
J_3 &= \int_0^t \int_\Omega f\cdot \dot{u}^\eta \, dx d\tau = -\int_\Omega f(0)u^\eta(0)dx + \int_\Omega f(t)u^\eta(t) + \int_0^t \int_\Omega \dot{f}u^\eta \, dx d\tau \\
&\leq C(t) + \frac{1}{2}\mathcal{E}(t) + \int_0^t \mathcal{E}(t) d\tau.
\end{aligned} \tag{20}
$$

Note that $u^\eta(0)$ and $A^\eta(0)$ are solutions of system of equations (2) and do not depend on $\eta$. Thus, C(t) in the last inequality and $\mathcal{E}(0)$ do not depend on $\eta$.

Putting inequalities (18), (19) and (20) into (17,) we obtain

$$
\mathcal{E}(t) \leq \frac{1}{2}\mathcal{E}(t) + C\int_0^t \mathcal{E} d\tau + C(t).
$$

This immediately gives

$$
\mathcal{E}(t) \leq C(t) + C\int_0^t \mathcal{E} d\tau.
$$

Finally, the Gronwall inequality completes the proof. $\square$

Directly from Theorem 8 we get that $y^\eta$ is bounded in $L^\infty((0,T),L^2(\Omega))$. Additionally, Theorem 7 yields that $u^\eta$ is bounded in $L^\infty((0,T),H^1(\Omega,\mathbb{R}^3))$ and $A^\eta$ is bounded in $L^\infty((0,T),H^1(\Omega,\mathfrak{so}(3)))$. The tensor $\varepsilon_p^\eta$ is bounded in $L^\infty((0,T),L^2(\Omega,\operatorname{Sym}(3)))$ because of the inequality

$$
|\varepsilon_p^\eta|^2 \leq 2\left(|\varepsilon^\eta|^2 + |\varepsilon^\eta - \varepsilon_p^\eta|^2\right) \leq C\mathcal{E}(u^\eta,\varepsilon^\eta,\varepsilon_p^\eta,A^\eta,y^\eta).
$$

Thus, we can find the subsequence (still denoted by $\eta$) such that we have

$$
\begin{aligned}
u^\eta &\overset{*}{\rightharpoonup} u && \text{in } L^\infty((0,T),H^1(\Omega,\mathbb{R}^3)),\\
A^\eta &\overset{*}{\rightharpoonup} A && \text{in } L^\infty((0,T),H^1(\Omega,\mathfrak{so}(3))),\\
\varepsilon_p^\eta &\overset{*}{\rightharpoonup} \varepsilon_p && \text{in } L^\infty((0,T),L^2(\Omega,\mathrm{Sym}(3))),\\
y^\eta &\overset{*}{\rightharpoonup} y && \text{in } L^\infty((0,T),L^2(\Omega)).
\end{aligned}
$$

Hence, the limit functions satisfy

$$
\begin{aligned}
\operatorname{div}\sigma &= -f,\\
\sigma &= 2\mu(\varepsilon-\varepsilon_p)+2\mu_c(\operatorname{skew}(\nabla u)-A)+\lambda\operatorname{tr}[\varepsilon]\cdot\mathbb{I},\\
-l_c\Delta\operatorname{axl}(A) &= \mu_c\operatorname{axl}(\operatorname{skew}(\nabla u)-A),\\
u|_{\partial\Omega} &= u_d,\ A|_{\partial\Omega}=A_d.
\end{aligned}
$$

In order to complete the proof of the existence of solution of system (1), we must show that equation (1d) holds.

**Theorem 9.** *Let us assume that*

$$
\begin{aligned}
f &\in C^2([0,\,T],\,L^2(\Omega,\,\mathbb{R}^3)), \qquad u_d\in C^3([0,\,T],\,H^{\frac{1}{2}}(\Omega,\,\mathbb{R}^3)),\\
A_d &\in C^3([0,\,T],\,H^{\frac{3}{2}}(\Omega,\,\mathbb{R}^3)),
\end{aligned}
$$

*and that $\varepsilon_p^0\in L^2(\Omega,\mathrm{Sym}(3))$ and $y^0\in L^2(\Omega)$ are such that*

$$
\begin{aligned}
F\left(2\mu(\varepsilon(0)-\varepsilon_p^0),-\tfrac{\gamma}{\alpha}y^0\right) &\in L^2(\Omega,\mathrm{Sym}(3)),\\
g\left(2\mu(\varepsilon(u(0))-\varepsilon_p^0),-\tfrac{\gamma}{\alpha}y^0\right) &\in L^2(\Omega).
\end{aligned}
\tag{21}
$$

*Then there exists a constant $C>0$ such that for all $\eta>0$ the following inequality*

$$
\mathcal{E}(\dot{u^\eta},\dot{\varepsilon^\eta},\dot{\varepsilon_p^\eta},\dot{A^\eta},\dot{y^\eta})(t)\le C \text{ for all } 0\le t\le T
$$

*is satisfied.*

*Proof.* We denote $u^\eta(t+h),\ \varepsilon^\eta(t+h),\ \varepsilon_p^\eta(t+h),\ A^\eta(t+h),\ y^\eta(t+h)$ by $u_h^\eta(t),\ \varepsilon_h^\eta(t),\ \varepsilon_{p,h}^\eta(t),\ A_h^\eta(t),\ y_h^\eta(t)$ respectively for a sufficiently small $h>0$. We calculate the derivative of the energy

$$
\begin{aligned}
&\dot{\mathcal{E}}(u_h^\eta-u^\eta,\varepsilon_h^\eta-\varepsilon^\eta,\varepsilon_{p,h}^\eta-\varepsilon_p^\eta,A_h^\eta-A^\eta,y_h^\eta-y^\eta)(t)\\
&\quad=\int_\Omega 2\mu(\varepsilon_h^\eta-\varepsilon^\eta-(\varepsilon_{p,h}^\eta-\varepsilon_p^\eta))\cdot(\dot\varepsilon_h^\eta-\dot\varepsilon^\eta-(\dot\varepsilon_{p,h}^\eta-\dot\varepsilon_p^\eta))\\
&\qquad+2\mu_c(\operatorname{skew}(\nabla u_h^\eta-\nabla u^\eta)-(A_h^\eta-A^\eta))\cdot(\operatorname{skew}(\nabla\dot u_h^\eta-\nabla\dot u^\eta)-(\dot A_h^\eta-\dot A^\eta))\\
&\qquad+4l_c\nabla\operatorname{axl}(A_h^\eta-A^\eta)\cdot\nabla\operatorname{axl}(\dot A_h^\eta-\dot A^\eta)+\frac{\gamma}{\alpha}(y_h^\eta-y^\eta)\cdot(\dot y_h^\eta-\dot y^\eta)\\
&\qquad+\lambda\operatorname{tr}[\varepsilon_h^\eta-\varepsilon^\eta]\operatorname{tr}[\dot\varepsilon_h^\eta-\dot\varepsilon^\eta]dx.
\end{aligned}
$$

We group terms

$$
\begin{aligned}
&\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta, y_h^\eta - y^\eta)(t) \\
&\quad = \int_\Omega (\sigma_h^\eta - \sigma^\eta) \cdot (\nabla \dot{u}_h^\eta - \nabla \dot{u}^\eta) dx \\
&\quad\quad - \int_\Omega ((T_{E,h}^\eta - T_E^\eta) \cdot (\dot{\varepsilon}_{p,h}^\eta - \dot{\varepsilon}_p^\eta) - \frac{\gamma}{\alpha}(y_h^\eta - y^\eta)(\dot{y}_h^\eta - \dot{y}^\eta)) dx \\
&\quad\quad + \int_\Omega 4 l_c \nabla \operatorname{axl}(A_h^\eta - A^\eta) \cdot \nabla \operatorname{axl}(\dot{A}_h^\eta - \dot{A}^\eta) dx \\
&\quad\quad - \int_\Omega 4\mu_c \operatorname{axl}(\operatorname{skew}(\nabla u_h^\eta - \nabla u^\eta) - (A_h^\eta - A^\eta)) \cdot \operatorname{axl}(\dot{A}_h^\eta - \dot{A}^\eta) dx,
\end{aligned}
$$

where $\sigma_h^\eta(t) = \sigma^\eta(t+h)$ and $T_{E,h}^\eta = 2\mu(\varepsilon_h^\eta - \varepsilon_{p,h}^\eta)$.

In the similar way as in the proof of Theorem 8 we obtain

$$
\begin{aligned}
&\dot{\mathcal{E}}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta, y_h^\eta - y^\eta)(t) \\
&\quad \le \int_\Omega (f_h - f) \cdot (\dot{u}_h^\eta - \dot{u}^\eta) dx + \int_{\partial\Omega} ((\sigma_h^\eta - \sigma^\eta) \cdot n) \cdot (\dot{u}_{d,h} - \dot{u}_d) dS \\
&\quad\quad + 4 l_c \int_{\partial\Omega} ((\nabla \operatorname{axl}(A_h^\eta - A^\eta)) \cdot n) \cdot \operatorname{axl}(\dot{A}_{d,h} - \dot{A}_d) dS,
\end{aligned}
$$

$f_h(t) = f(t+h)$, $u_{d,h}(t) = u(t+h)$ and $A_{d,h}(t) = A_d(t+h)$. Integrating the above inequality over $[0,t]$ we obtain

$$
\begin{aligned}
&\mathcal{E}(u_h^\eta - u^\eta, \varepsilon_h^\eta - \varepsilon^\eta, \varepsilon_{p,h}^\eta - \varepsilon_p^\eta, A_h^\eta - A^\eta, y_h^\eta - y^\eta)(t) \\
&\quad \le \mathcal{E}(0) + \int_0^t \int_\Omega (f_h - f) \cdot (\dot{u}_h^\eta - \dot{u}^\eta) dx\, d\tau + \int_0^t \int_{\partial\Omega} ((\sigma_h^\eta - \sigma^\eta) \cdot n) \cdot (\dot{u}_{d,h} - \dot{u}_d) dS\, d\tau \\
&\quad\quad + \int_0^t 4 l_c \int_{\partial\Omega} ((\nabla \operatorname{axl}(A_h^\eta - A^\eta)) \cdot n) \cdot \operatorname{axl}(\dot{A}_{d,h} - \dot{A}_d) dS\, d\tau.
\end{aligned} \tag{22}
$$

The first integral on the right-hand side of the above inequality is calculated as follows

$$
\begin{aligned}
&\int_0^t \int_\Omega (f_h - f) \cdot (\dot{u}_h^\eta - \dot{u}^\eta) dx d\tau \\
&\qquad = \int_0^t \int_\Omega (f_h - f) \cdot \dot{u}_h^\eta dx\, d\tau - \int_0^t \int_\Omega (f_h - f) \cdot \dot{u}^\eta dx\, d\tau \\
&\qquad = \int_h^{t+h} \int_\Omega (f - f_{-h}) \cdot \dot{u}^\eta dx\, d\tau - \int_0^t \int_\Omega (f_h - f) \cdot \dot{u}^\eta dx\, d\tau \\
&\qquad = \int_0^t \int_\Omega (2f - f_h - f_{-h}) \cdot \dot{u}^\eta dx\, d\tau + \int_t^{t+h} (f - f_{-h}) \cdot \dot{u}^\eta dx\, d\tau \\
&\qquad\quad - \int_0^h \int_\Omega (f - f_{-h}) \cdot \dot{u}^\eta dx d\tau.
\end{aligned}
$$

In the analogous way we proceed with the other integrals in inequality (22). Then, we divide both sides of the obtained inequality by $h^2$ and pass to the limit as $h \to 0^+$. Hence, we get

$$\begin{aligned}\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(t) \leq \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(0) + \int_0^t \int_\Omega \ddot{f} \cdot \dot{u}^\eta dx d\tau + \int_\Omega \dot{f} \cdot \dot{u}^\eta dx \\ - \int_\Omega \dot{f}(0) \cdot \dot{u}^\eta(0) dx + \int_0^t \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \dddot{u}_d dS \\ + \int_{\partial\Omega} (\sigma^\eta \cdot n) \cdot \ddot{u}_d dS - \int_{\partial\Omega} (\sigma^\eta(0) \cdot n) \cdot \ddot{u}_d(0) dS \\ + 4l_c \int_0^t (\nabla \operatorname{axl}(A^\eta) \cdot n) \cdot \operatorname{axl}(\dddot{A}_d) dS d\tau \\ + 4l_c \int_{\partial\Omega} (\nabla axl(A^\eta) \cdot n) \cdot \operatorname{axl}(\ddot{A}_d) dS \\ - 4l_c \int_{\partial\Omega} (\nabla \operatorname{axl}(A^\eta(0)) \cdot n) \cdot \operatorname{axl}(\ddot{A}_d(0)) dS. \end{aligned} \tag{23}$$

The functions $\dot{u}^\eta(0)$ and $\dot{A}^\eta(0)$ are the solution of the system

$$\begin{aligned}\operatorname{div} \dot{\sigma}^\eta &= -\dot{f}(0), \\ \dot{\sigma}^\eta &= 2\mu \left(\varepsilon(\dot{u}_0^\eta) - F^\eta(2\mu(\varepsilon(0) - \varepsilon_p^0), -\tfrac{\gamma}{\alpha} y_0)\right) \\ &\quad + 2\mu_c \left(\operatorname{skew}(\nabla \dot{u}^\eta(0)) - \dot{A}^\eta(0)\right) + \lambda \operatorname{tr}[\varepsilon(\dot{u}^\eta(0))]\mathbb{I}, \\ -l_c \Delta \operatorname{axl}\left(\dot{A}^\eta(0)\right) &= \mu_c \operatorname{axl}(\operatorname{skew}(\nabla \dot{u}^\eta(0)) - \dot{A}^\eta(0)), \\ \dot{u}^\eta(0)|_{\partial\Omega} &= \dot{u}_d(0), \; \dot{A}^\eta(0)|_{\partial\Omega} = \dot{A}_d(0).\end{aligned}$$

The sequence $\dot{\varepsilon}_p^\eta(0) = F^\eta(2\mu(\varepsilon(0) - \varepsilon_p^0), -\frac{\gamma}{\alpha} y^0)$ is bounded in $L^2(\Omega, \mathrm{Sym}(3))$ and the sequence $\dot{y}^\eta(0) = g^\eta(2\mu(\varepsilon(0) - \varepsilon_p^0), -\frac{\gamma}{\alpha} y^0)$ is bounded in $L^2(\Omega)$ because of assumptions (21) and a property of the Yosida approximation. Therefore, the sequences $\dot{u}^\eta(0)$ and $\dot{A}^\eta(0)$ are bounded in their respective spaces and also $\dot{\varepsilon}_p^\eta(0)$ and $\dot{y}^\eta(0)$ are bounded. Thus, the sequence

$$\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(0)$$

is bounded independently of $\eta$.

We bound the terms on the right-hand side of inequality (23) in the analogous manner as in the proof of Theorem 8. Hence, we arrive at the following inequality

$$\begin{aligned}\mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(t) \leq \frac{1}{2} \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(t) \\ + \int_0^t \mathcal{E}(\dot{u}^\eta, \dot{\varepsilon}^\eta, \dot{\varepsilon}_p^\eta, \dot{A}^\eta, \dot{y}^\eta)(\tau) d\tau + C(t).\end{aligned}$$

The proof is finished due to Gronwall's inequality. □

The sequences $\dot{\varepsilon}_p^\eta = F^\eta(2\mu(\varepsilon^\eta - \varepsilon_p^\eta), -\frac{\gamma}{\alpha} y^\eta)$ and $\dot{y}^\eta = g^\eta(2\mu(\varepsilon^\eta - \varepsilon_p^\eta), -\frac{\gamma}{\alpha} y^\eta)$ are bounded in the spaces $L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3)))$ and $L^\infty((0,T), L^(\Omega))$ respectively. Thus, we have a subsequence such that a *-weak convergence

$$\begin{aligned}\dot{\varepsilon}_p^\eta &= F^\eta(T_E^\eta, \, -\tfrac{\gamma}{\alpha} y^\eta) \overset{*}{\rightharpoonup} \dot{\varepsilon}_p \quad \text{in } L^\infty((0,T), L^2(\Omega, \mathrm{Sym}(3))), \\ \dot{y}^\eta &= g^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha} y^\eta) \overset{*}{\rightharpoonup} \dot{y} \quad \text{in } L^\infty((0, \, T), \, L^2(\Omega))\end{aligned} \tag{24}$$

holds. In order to complete the proof, we must show that equalities $\dot{\varepsilon}_p = F(T_E,\ -\frac{\gamma}{\alpha}y)$ and $\dot{y} = g(T_E,\ -\frac{\gamma}{\alpha}y)$ are satisfied. We need a stronger convergence here.

**Theorem 10.** *Under the hypotheses of Theorem 9 the convergence*

$$\mathcal{E}(u^\eta - u^\nu, \varepsilon^\eta - \varepsilon^\nu, \varepsilon_p^\eta - \varepsilon_p^\nu, A^\eta - A^\nu, y^\eta - y^\nu)(t) \to 0 \tag{25}$$

*for $\nu, \eta \to 0^+$ holds uniformly on $[0,T]$.*

*Proof.* Calculating the derivative of the energy from (25) we obtain

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\nu - u^\eta, \varepsilon^\nu - \varepsilon^\eta, \varepsilon_p^\nu - \varepsilon_p^\eta, A^\nu - A^\eta, y^\nu - y^\eta)(t)\\
&\quad = \int_\Omega 2\mu(\varepsilon^\nu - \varepsilon^\eta - \varepsilon_p^\nu + \varepsilon_p^\eta)\cdot(\dot{\varepsilon}^\nu - \dot{\varepsilon}^\eta - \dot{\varepsilon}_p^\nu + \dot{\varepsilon}_p^\eta) + \lambda\,\mathrm{tr}[\varepsilon^\nu - \varepsilon^\eta]\,\mathrm{tr}[\dot{\varepsilon}^\nu - \dot{\varepsilon}^\eta]\\
&\qquad + 2\mu_c\,(\mathrm{skew}(\nabla u^\nu - \nabla u^\eta) - A^\nu + A^\eta)\cdot(\mathrm{skew}(\nabla \dot{u}^\nu - \nabla \dot{u}^\eta) - \dot{A}^\nu + \dot{A}^\eta)\\
&\qquad + 4l_c(\nabla\,\mathrm{axl}A^\nu - \nabla\,\mathrm{axl}A^\eta)\cdot(\nabla\,\mathrm{axl}\dot{A}^\nu - \nabla\,\mathrm{axl}\dot{A}^\eta)dx.
\end{aligned}$$

Because both approximation steps have the same boundary values we conclude

$$\begin{aligned}
&\dot{\mathcal{E}}(u^\nu - u^\eta,\ \varepsilon^\nu - \varepsilon^\eta,\ \varepsilon_p^\nu - \varepsilon_p^\eta,\ A^\nu - A^\eta,\ y^\nu - y^\eta)(t)\\
&\quad = \int_\Omega -(T_E^\nu - T_E^\eta)\cdot(\dot{\varepsilon}_p^\nu - \dot{\varepsilon}_p^\eta) + \tfrac{\gamma}{\alpha}(y^\nu - y^\eta)\cdot(\dot{y}^\nu - \dot{y}^\eta)dx.
\end{aligned}$$

Let $(J^\theta, j^\theta)$ be the resolvent of $(F,g)$ for $\theta > 0$. Then, from the definition we have

$$\theta F^\theta + J^\theta = I \text{ and } \theta g^\theta + j^\theta = I.$$

The following equalites

$$F^\theta(x) = F(J^\theta(x), j^\theta(x)),\ g^\theta(x) = g(J^\theta(x), j^\theta(x)) \text{ for all } x \in \mathrm{Sym}(3)\times\mathbb{R}$$

are also satisfied. Thus, thanks to (6d) we obtain

$$\begin{aligned}
&\int_\Omega -(T_E^\nu - T_E^\eta)\cdot(\dot{\varepsilon}_p^\nu - \dot{\varepsilon}_p^\eta) + \tfrac{\gamma}{\alpha}(y^\nu - y^\eta)\cdot(\dot{y}^\nu - \dot{y}^\eta)dx\\
&\quad = \int_\Omega -\big((J^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu) - J^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta)\big)\\
&\qquad\qquad \cdot\Big(F\big(J^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu), j^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu)\big) - F\big(J^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta), j^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta)\big)\Big)dx\\
&\qquad - \int_\Omega \big((j^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu) - j^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta)\big)\\
&\qquad\qquad \cdot\Big(g\big(J^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu), j^\nu(T_E^\nu, -\tfrac{\gamma}{\alpha}y^\nu)\big) - g\big(J^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta), j^\eta(T_E^\eta, -\tfrac{\gamma}{\alpha}y^\eta)\big)\Big)dx\\
&\qquad + \int_\Omega -(\nu\dot{\varepsilon}_p^\nu - \eta\dot{\varepsilon}_p^\eta)\cdot(\dot{\varepsilon}_p^\nu - \dot{\varepsilon}_p^\eta) - (\nu\dot{y}^\nu - \eta\dot{y}^\eta)\cdot(\dot{y}^\nu - \dot{y}^\eta)dx\\
&\quad \le \int_\Omega (\nu + \eta)\cdot(\dot{\varepsilon}_p^\eta\cdot\dot{\varepsilon}_p^\eta + \dot{y}^\eta\cdot\dot{y}^\nu)dx \le\\
&\quad \le (\nu + \eta)(\|\dot{\varepsilon}_p^\eta\|_{L^2(\Omega)}\|\dot{\varepsilon}_p^\nu\|_{L^2(\Omega)} + \|\dot{y}^\eta\|_{L^2(\Omega)}\|\dot{y}^\nu\|_{L^2(\Omega)}).
\end{aligned}$$

Norms on the right-hand side of the above inequality are bounded in virtue of Theorem 9. Thus, we conclude

$$\dot{\mathcal{E}}(u^\nu - u^\eta,\ \varepsilon^\nu - \varepsilon^\eta,\ \varepsilon_p^\nu - \varepsilon_p^\eta,\ A^\nu - A^\eta,\ y^\nu - y^\eta)(t) \le C(\nu+\eta).$$

We integrate this inequality by time and get $\mathcal{E}(t) \le C(T)(\nu+\eta)$. □

## 4.3. PROOF OF WELL-POSEDNESS

Finally, we prove the main theorem of the article.

*Proof of Theorem 1.* In virtue of Theorem 10 we obtain

$$\begin{aligned} &T_E^\eta \to T_E \quad \text{in } L^\infty((0,T),L^2(\Omega,\mathrm{Sym}(3))),\\ &y^\eta \to y \quad\ \ \text{in } L^\infty((0,T),L^2(\Omega)). \end{aligned} \tag{26}$$

Because $(J^\eta, j^\eta)$ is a Lipschitz function with a constant 1, so we arrive at the following inequality

$$\begin{aligned} &\left|\left(J^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta), j^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta)\right) - (T_E,\ -\tfrac{\gamma}{\alpha}y)\right|\\ &\quad\le \left|\left(J^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta), j^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta)\right) - \left(J^\eta(T_E,-\tfrac{\gamma}{\alpha}y), j^\eta(T_E,-\tfrac{\gamma}{\alpha}y)\right)\right|\\ &\qquad + \left|\left(J^\eta(T_E,-\tfrac{\gamma}{\alpha}y), j^\eta(T_E,-\tfrac{\gamma}{\alpha}y)\right) - (T_E,-\tfrac{\gamma}{\alpha}y)\right|\\ &\quad\le |(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta) - (T_E,-\tfrac{\gamma}{\alpha}y)| + |(J^\eta(T_E,-\frac{\gamma}{\alpha}y), j^\eta(T_E,-\tfrac{\gamma}{\alpha}y)) - (T_E,-\tfrac{\gamma}{\alpha}y)|. \end{aligned}$$

Thus, we conclude that

$$\left(J^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta), j^\eta(T_E^\eta,-\tfrac{\gamma}{\alpha}y^\eta)\right) \to (T_E,\ -\tfrac{\gamma}{\alpha}y).$$

This and (24) and (26) yield that the equations

$$\begin{aligned} \dot{\varepsilon}_p &= F(T_E,-\tfrac{\gamma}{\alpha}y),\\ \dot{y} &= g(T_E,-\tfrac{\gamma}{\alpha}y) \end{aligned}$$

are satisfied. Therefore, we have the existence of solutions of (1). We must still show their uniqueness.

Let $(u^1,\varepsilon_p^1,A^1,y^1)$ and $(u^2,\varepsilon_p^2,A^2,y^2)$ be solutions of (1). We evaluate the energy on the difference of these solutions

$$\begin{aligned} &\dot{\mathcal{E}}(u^1-u^2,\varepsilon^1-\varepsilon^2,\varepsilon_p^1-\varepsilon_p^2,A^1-A^2,y^1-y^2)(t)\\ &\quad= \int_\Omega 2\mu(\varepsilon^1-\varepsilon^2-\varepsilon_p^1+\varepsilon_p^2)\cdot(\dot{\varepsilon}^1-\dot{\varepsilon}^2-\dot{\varepsilon}_p^1+\dot{\varepsilon}_p^2) + \lambda\,\mathrm{tr}[\varepsilon^1-\varepsilon^2]\cdot\mathrm{tr}[\dot{\varepsilon}^1-\dot{\varepsilon}^2]\\ &\qquad + 2\mu_c(\mathrm{skew}(\nabla(u^1-u^2)) - A^1 + A^2)\cdot(\mathrm{skew}(\nabla(\dot{u}^1-\dot{u}^2)) - \dot{A}^1+\dot{A}^2)\\ &\qquad + 4l_c\nabla(\mathrm{axl}(A^1-A^2))\cdot\nabla(\mathrm{axl}(\dot{A}^1-\dot{A}^2)) + \tfrac{\gamma}{\alpha}(y^1-y^2)\cdot(\dot{y}^1-\dot{y}^2)dx\\ &\quad= \int_\Omega -(T_E^1-T_E^2)\cdot(\dot{\varepsilon}_p^1-\dot{\varepsilon}_p^2) + (\tfrac{\gamma}{\alpha}y^1 - \tfrac{\gamma}{\alpha}y^2)\cdot(\dot{y^1}-\dot{y^2})dx \end{aligned}$$

The last integral in the above inequality is non positive because of monotinicity of $(F,g)$. Hence, we obtain

$$\dot{\mathcal{E}}(u^1-u^2,\varepsilon^1-\varepsilon^2,\varepsilon_p^1-\varepsilon_p^2,A^1-A^2,y^1-y^2)(t)\leq 0.$$

Thus, $\mathcal{E}$ is non-increasing. Because $\mathcal{E}(0)=0$, we have $\mathcal{E}(t)=0$ for $0\leq t\leq T$. This finishes the proof. □


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# QUATERNIONIC COMPRESSED SENSING

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**Abstract:** The article concerns compressed sensing methods in the quaternion algebra. We prove that it is possible to uniquely reconstruct – by $\ell_1$-norm minimization – a sparse quaternion signal from a limited number of its linear measurements, provided the quaternion measurement matrix satisfies the so-called restricted isometry property with a sufficiently small constant. We provide error estimates for the reconstruction of a non-sparse quaternion signal in the noisy and noiseless cases. We also prove that quaternion Gaussian random matrices satisfy the restricted isometry property with overwhelming probability.
**Keywords:** compressed sensing, quaternion, restricted isometry property
**Mathematics Subject Classification (2020):** 46S10 (primary), 65J10, 60B20

## 1. INTRODUCTION

E. Candès et al. showed that – in the real or complex setting – if a measurement matrix satisfies the so-called *restricted isometry property* (RIP) with a sufficiently small constant, then every sparse signal can be uniquely reconstructed from a limited number of its linear measurements as a solution of a convex program of $\ell_1$-norm minimization (see e.g. [6, 7, 8, 13] for more references). The sparsity of the signal is a natural assumption – most of the well known signals have a sparse representation in the appropriate basis (e.g. the wavelet representation of an image). Moreover, if the original signal was not sparse, the same minimization procedure provides a good sparse approximation of the signal and the procedure is stable in the sense that the error is bounded above by the $\ell_1$-norm of the difference between the original signal and its best sparse approximation.

For a certain time, the attention of the researchers in the theory of *compressed sensing* had mostly been focused on the real and complex signals. Over the last decade, results of numerical experiments have been published, suggesting that the compressed sensing methods can be successfully applied also in the quaternion algebra [2, 17, 20, 30]. However, until recently there were no theoretical results that could explain the success of these experiments. The aim of our research is to develop theoretical background of the compressed sensing theory in the quaternion algebra.

Our first step towards this goal is proving that one can uniquely reconstruct a sparse quaternion signal – by $\ell_1$-norm minimization – provided the real measurement matrix satisfies the RIP (for quaternion vectors) with a sufficiently small constant ([1, Corrolary 5.1]). This result can be directly applied, since any real matrix satisfying the RIP for real vectors satisfies the RIP for quaternion vectors with the same constant ([1, Lemma 3.2]). We also want to point out a very interesting recent result by N. Gomes, S. Hartmann and U. Kähler concerning the quaternion Fourier matrices arising in colour representation of images. They showed that with high probability such matrices allow a sparse reconstruction by means of the $\ell_1$-norm minimization [15, Theorem 3.2]. Their proof, however, is straightforward and does not use the notion of RIP.

The generalization of compressed sensing to the quaternion algebra would be significant due to their wide applications. Apart from the classical applications (in quantum mechanics and for the description of 3D solid body rotations), quaternions have also been used in 3D and 4D signal processing [24], in particular to represent colour images (e.g. in the RGB or CMYK models). Due to the extension of classical tools (like the Fourier transform [11]) to the quaternion algebra, it is possible to investigate colour images without the need of treating each component separately [10, 11]. That is why quaternions have found numerous applications in image filtering, image enhancement, pattern recognition, edge detection and watermarking [12, 14, 19, 21, 23, 26, 29]. A dual-tree quaternion wavelet transform in a multiscale analysis of geometric image features has also been proposed [9]. For this purpose, an alternative representation of quaternions is used – through its magnitude (norm) and three phase angles: two of them encode phase shifts, while the third one contains image texture information [4]. In view of numerous articles presenting results of numerical experiments of quaternion signal processing and their possible applications, there is a natural need of further thorough theoretical investigations in this field.

In the first part of this article, we extend the fundamental result of the compressed sensing theory to the quaternion case. Namely, we show that if a quaternion measurement matrix satisfies the RIP with a sufficiently small constant, then it is possible to reconstruct sparse quaternion signals from a small number of their measurements via $\ell_1$-norm minimization (Corollary 6). We also estimate the error of reconstruction of a non-sparse signal from exact and noisy data (Theorem 5). Note that these results not only generalize the previous ones [1, Theorem 4.1, Corrolary 5.1], but also improve them by decreasing the error estimation's constants. This enhancement was possible due to the fact that algebraic properties of quaternion Hermitian matrices (Lemma 1) were used to derive characterization of the restricted isometry constants (Lemma 3) analogous to the real and complex case. Consequently, one can carefully follow the steps of the classical Candès' proof [6] with caution to the non-commutativity of quaternion multiplication.

Furthermore, it is known that real Gaussian and Bernoulli random matrices and partial Discrete Fourier Transform matrices, to name just a few, satisfy the RIP (with overwhelming probability) [13]. However, until recently there were no known examples of quaternion matrices satisfying this condition. It has been believed that quaternion Gaussian random matrices satisfy RIP and, therefore, they have been widely used in numerical experiments [2, 17, 30], but there was a lack of theoretical justification of this conviction. In the second part of this article, we prove that this hypothesis is true, i.e. *quaternion Gaussian random matrices satisfy the RIP*, and we provide estimates on matrix sizes that guarantee the RIP with overwhelming probability (Theorem 11). The existence of quaternion matrices satisfying the RIP, together with the main results of this article (Theorem 5, Corollary 6), constitute the theoretical foundation of the classical compressed sensing methods in the quaternion algebra already used in color image processing [2, 16, 20].

The article is organized as follows: in section 2 we recall basic notation and facts concerning the quaternion algebra with particular emphasis on the properties of Hermitian form and Hermitian matrices. The third section is devoted to the RIP and characterization of the restricted isometry constants in terms of Hermitian matrix norm. In the fourth and fifth sections, we present proofs of the fundamental results of the compressed sensing theory in the quaternion case. The sixth section is dedicated to quaternion random variables and matrices. We define the quaternion Gaussian random variable with mean zero and variance $\sigma^2$, denoted by $X \sim \mathcal{N}_{\mathbb{H}}\left(0, \sigma^2\right)$, in particular we always assume independence of its components – note that this aspect was not clear in [30]. We also provide distribution of the Rayleigh quotient $\mathcal{R}$ for quaternion Gaussian random matrices and verify that it is sub-exponential with appropriate parameters. In the seventh section, we prove the RIP for quaternion Gaussian random matrices. Finally, in the eighth section we present results of numerical experiments illustrating our results – we may see, in particular, that the rate of perfect reconstructions in the quaternion case is higher than in the real case experiments with the same parameters. We conclude with a short résumé of the obtained results and our further research perspectives.

## 2. ALGEBRA OF QUATERNIONS

Denote by $\mathbb{H}$ the algebra of quaternions

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad \text{where} \quad a, b, c, d \in \mathbb{R},$$

endowed with the standard norm

$$|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2},$$

where $\overline{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ is the conjugate of $q$. Recall that multiplication is associative but in general not commutative in the quaternion algebra and is defined by the following rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

and

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

Multiplication is distributive over addition and has a neutral element $1 \in \mathbb{H}$ hence, $\mathbb{H}$ forms a ring which is usually called a *noncommutative field*. We also have

$$\overline{q \cdot w} = \overline{w} \cdot \overline{q} \quad \text{for any} \quad q, w \in \mathbb{H}.$$

In what follows, we will interpret signals as vectors with quaternion coordinates, i.e., elements of $\mathbb{H}^n$. Algebraically, $\mathbb{H}^n$ is a module over the ring $\mathbb{H}$, usually called the *quaternion vector space* (although it is not really a vector space since $\mathbb{H}$ is not a field). We will also consider matrices with quaternion entries and with usual multiplication rules.

For any matrix $\Phi \in \mathbb{H}^{m\times n}$ with quaternion entries by $\Phi^*$ we denote the adjoint matrix, i.e., $\Phi^* = \overline{\Phi}^T$, where $T$ is the transpose. The same notation applies to quaternion vectors $\mathbf{x} \in \mathbb{H}^n$ which can be interpreted as one-column matrices $\mathbf{x} \in \mathbb{H}^{n\times 1}$. Obviously, $(\Phi^*)^* = \Phi$.

A matrix $\Phi \in \mathbb{H}^{m\times n}$ defines an $\mathbb{H}$-linear transformation $\Phi : \mathbb{H}^n \to \mathbb{H}^m$ (in terms of the right quaternion vector space, i.e., considering the right scalar multiplication) which behaves as the standard matrix-vector multiplication:

$$\Phi(\mathbf{x}+\mathbf{y}) = \Phi\mathbf{x} + \Phi\mathbf{y} \quad \text{and} \quad \Phi(\mathbf{x}q) = (\Phi\mathbf{x})q \quad \text{for any} \quad \mathbf{x}, \mathbf{y} \in \mathbb{H}^n,\ q \in \mathbb{H}.$$

We also have

$$(\Phi q)^* = \overline{q}\Phi^*, \quad (q\Phi)^* = \Phi^*\overline{q}, \quad (\Phi\mathbf{x})^* = \mathbf{x}^*\Phi^*, \quad (\Phi\Psi)^* = \Psi^*\Phi^*$$

for all $\Phi \in \mathbb{H}^{m\times n}$, $q \in \mathbb{H}$, $\mathbf{x} \in \mathbb{H}^n$, $\Psi \in \mathbb{H}^{n\times p}$.

For any $n \in \mathbb{N}$, we introduce the following Hermitian form $\langle\cdot,\cdot\rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ with quaternion values:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^*\mathbf{x} = \sum_{i=1}^{n} \overline{y_i}x_i, \quad \text{where} \quad \mathbf{x} = (x_1, \ldots, x_n)^T,\ \mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{H}^n$$

where $T$ is the transpose. We also denote

$$\|\mathbf{x}\|_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} \quad \text{for any } \mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{H}^n.$$

It is a straightforward calculation to verify that $\langle\cdot,\cdot\rangle$ satisfies all properties of an inner product, i.e., for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{H}^n$ and $q \in \mathbb{H}$:

- $\overline{\langle \mathbf{x}, \mathbf{y} \rangle} = \langle \mathbf{y}, \mathbf{x} \rangle$;
- $\langle \mathbf{x}q, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle q$;
- $\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$;
- $\langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|_2^2 \geq 0$ and $\|\mathbf{x}\|_2^2 = 0 \iff \mathbf{x} = \mathbf{0}$.

Hence, $\|\cdot\|_2$ satisfies the axioms of a norm in $\mathbb{H}^n$.

By carefully following the steps of the classical proof we also get the Cauchy-Schwarz inequality (cf. [1, Lemma 2.2]):

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{H}^n$.

Notice that for $\Phi \in \mathbb{H}^{m \times n}$, the matrix $\Phi^*$ defines the adjoint $\mathbb{H}$-linear transformation because

$$\langle \mathbf{x}, \Phi^* \mathbf{y} \rangle = (\Phi^* \mathbf{y})^* \mathbf{x} = \mathbf{y}^* \Phi \mathbf{x} = \langle \Phi \mathbf{x}, \mathbf{y} \rangle \quad \text{for} \quad \mathbf{x} \in \mathbb{H}^n, \mathbf{y} \in \mathbb{H}^m.$$

Recall that a linear transformation (matrix) $\Psi \in \mathbb{H}^{n \times n}$ is called *Hermitian* if $\Psi^* = \Psi$. Obviously, $\Phi^* \Phi$ is Hermitian for any $\Phi \in \mathbb{H}^{m \times n}$.

In the next section, we will use the following property of Hermitian matrices:

**Lemma 1.** *Suppose $\Psi \in \mathbb{H}^{n \times n}$ is Hermitian. Then*

$$\|\Psi\|_{2 \to 2} = \max_{\mathbf{x} \in \mathbb{H}^n, \|\mathbf{x}\|_2 = 1} |\langle \Psi \mathbf{x}, \mathbf{x} \rangle| = \max_{\mathbf{x} \in \mathbb{H}^n \setminus \{\mathbf{0}\}} \frac{|\langle \Psi \mathbf{x}, \mathbf{x} \rangle|}{\|\mathbf{x}\|_2^2},$$

*where $\|\cdot\|_{2 \to 2}$ is the standard operator norm in the right quaternion vector space $\mathbb{H}^n$ endowed with the norm $\|\cdot\|_2$, i.e.*

$$\|\Psi\|_{2 \to 2} = \max_{\mathbf{x} \in \mathbb{H}^n \setminus \{\mathbf{0}\}} \frac{\|\Psi \mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\mathbf{x} \in \mathbb{H}^n, \|\mathbf{x}\|_2 = 1} \|\Psi \mathbf{x}\|_2 .$$

*Proof.* Recall that a Hermitian matrix has real (right) eigenvalues [22]. Moreover, there exists an orthonormal (in terms of the $\mathbb{H}$-linear form $\langle \cdot, \cdot \rangle$) basis of $\mathbb{H}^n$ consisting of eigenvectors $\mathbf{x}_i$ corresponding to eigenvalues $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$ (cf. [22, Theorem 5.3.6. (c)]), i.e.,

$$\Psi \mathbf{x}_i = \mathbf{x}_i \lambda_i \quad \text{and} \quad \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \mathbf{x}_j^* \mathbf{x}_i = \delta_{i,j} \quad \text{for} \quad i, j = 1, \dots, n.$$

Denote $\lambda_{\max} = \max_{1 \leq i \leq n} |\lambda_i|$. We claim that $\|\Psi\|_{2 \to 2} = \lambda_{\max}$. Indeed, since the basis is orthonormal, for any vector $\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i \alpha_i \in \mathbb{H}^n$ such that $\|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} |\alpha_i|^2 = 1$, we have

$$\|\Psi \mathbf{x}\|_2^2 = \left\| \sum_{i=1}^{n} \Psi \mathbf{x}_i \alpha_i \right\|_2^2 = \left\| \sum_{i=1}^{n} \mathbf{x}_i \lambda_i \alpha_i \right\|_2^2 = \sum_{i=1}^{n} |\lambda_i \alpha_i|^2 \leq \lambda_{\max} \underbrace{\sum_{i=1}^{n} |\alpha_i|^2}_{=1}$$

and for $k$ such that $|\lambda_k| = \lambda_{\max}$,

$$\|\Psi \mathbf{x}_k\|_2 = \|\mathbf{x}_k \lambda_k\|_2 = \lambda_{\max} \|\mathbf{x}_k\|_2 = \lambda_{\max}.$$

On the other hand, since $\lambda_i$ are real,

$$\langle \Psi \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* \Psi \mathbf{x} = \left( \sum_{i=1}^{n} \mathbf{x}_i \alpha_i \right)^* \left( \sum_{j=1}^{n} \Psi \mathbf{x}_j \alpha_j \right) = \left( \sum_{i=1}^{n} \overline{\alpha_i} \mathbf{x}_i^* \right) \left( \sum_{j=1}^{n} \mathbf{x}_j \lambda_j \alpha_j \right)$$

$$= \sum_{i=1}^{n} \overline{\alpha_i}\lambda_i\alpha_i \overset{\lambda_i\in\mathbb{R}}{=} \sum_{i=1}^{n} \lambda_i \left|\alpha_i\right|^2.$$

Hence,

$$\left|\langle\Psi\mathbf{x},\mathbf{x}\rangle\right| \leq \lambda_{\max} \sum_{i=1}^{n} \left|\alpha_i\right|^2 = \lambda_{\max}$$

and – again – for the appropriate eigenvector the above quantities are equal. The result follows. □

In what follows, we will consider $\|\cdot\|_p$ norms for quaternion vectors $\mathbf{x} \in \mathbb{H}^n$ defined in the standard way:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{for} \quad p \in [1,\infty)$$

and

$$\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|,$$

where $\mathbf{x} = (x_1,\ldots,x_n)^T$. We will also apply the usual notation for the cardinality of the support of $\mathbf{x}$, i.e.

$$\|\mathbf{x}\|_0 = \#\mathrm{supp}(\mathbf{x}), \quad \text{where} \quad \mathrm{supp}(\mathbf{x}) = \{i \in \{1,\ldots,n\} : x_i \neq 0\}.$$

# 3. RESTRICTED ISOMETRY PROPERTY

Recall that a vector $\mathbf{x} \in \mathbb{H}^n$ is called $s$-sparse if it has at most $s$ nonzero coordinates, i.e.,

$$\|\mathbf{x}\|_0 \leq s.$$

As we mentioned in the introduction, one of the conditions which guarantee exact reconstruction of a sparse real signal from its few linear measurements is that the measurement matrix satisfies the so-called restricted isometry property (RIP) with a sufficiently small constant. The notion of restricted isometry constants was introduced by Candès and Tao in [8]. Here we generalize it to quaternion signals.

**Definition 2.** *Let $s \in \mathbb{N}$ and $\Phi \in \mathbb{H}^{m\times n}$. We say that $\Phi$ satisfies the s-restricted isometry property (for quaternion vectors) with a constant $\delta_s \geq 0$ if*

$$(1-\delta_s)\,\|\mathbf{x}\|_2^2 \leq \|\Phi\mathbf{x}\|_2^2 \leq (1+\delta_s)\,\|\mathbf{x}\|_2^2 \tag{1}$$

*for all s-sparse quaternion vectors $\mathbf{x} \in \mathbb{H}^n$. The smallest number $\delta_s \geq 0$ with this property is called the s-restricted isometry constant.*

Note that we can define $s$-restricted isometry constants for any matrix $\Phi \in \mathbb{H}^{m\times n}$ and any $s \in \{1,\ldots,n\}$. It has been proved that if a real matrix $\Phi \in \mathbb{R}^{m\times n}$ satisfies the inequality (1) for real $s$-sparse vectors $\mathbf{x} \in \mathbb{R}^n$, then it also satisfies it – with the same constant $\delta_s$ – for $s$-sparse quaternion vectors $\mathbf{x} \in \mathbb{H}^n$ [1, Lemma 3.2].

The following lemma extends an analogous result, known for real and complex matrices [13], to the quaternion case. For every matrix $\Phi \in \mathbb{H}^{m\times n}$ and for every $s$-element set of indices $S \subset \{1,\dots,n\}$ with $\#S = s$ by $\Phi_S \in \mathbb{H}^{m\times s}$ we denote the matrix consisting of columns of $\Phi$ with indices in $S$.

**Lemma 3.** *The s-restricted isometry constant of a matrix $\Phi \in \mathbb{H}^{m\times n}$ equals*

$$\max_{S\subset\{1,\dots,n\},\#S\leq s} \|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to 2}.$$

*Proof.* We proceed as in [13, Chapter 6]. Fix any $s \in \{1,\dots,n\}$ and $S \subset \{1,\dots,n\}$ with $\#S \leq s$. Notice that the condition (1) is equivalent to

$$\left|\|\Phi_S\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2\right| \leq \delta_s \|\mathbf{x}\|_2^2 \quad \text{for all} \quad \mathbf{x} \in \mathbb{H}^s,$$

where $\delta_s$ is the $s$-restricted isometry constant of $\Phi$. The left hand side equals

$$\left|\|\Phi_S\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2\right| = |\langle \Phi_S\mathbf{x}, \Phi_S\mathbf{x}\rangle - \langle \mathbf{x}, \mathbf{x}\rangle| = |\langle (\Phi_S^*\Phi_S - \mathbf{Id})\,\mathbf{x}, \mathbf{x}\rangle|$$

and, by Lemma 1, since the matrix $\Phi_S^*\Phi_S - \mathbf{Id}$ is Hermitian, we get

$$\max_{\mathbf{x}\in\mathbb{H}^s\setminus\{\mathbf{0}\}} \frac{|\langle (\Phi_S^*\Phi_S - \mathbf{Id})\,\mathbf{x}, \mathbf{x}\rangle|}{\|\mathbf{x}\|_2^2} = \|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to 2}.$$

□

The next result is an important tool in the proof of Theorem 5. From Lemma 3 and the Cauchy-Schwarz inequality we can obtain the same estimate as in the real and complex cases (cf. [6, Lemma 2.1] and [13, Proposition 6.3]) for quaternion vectors.

**Lemma 4.** *Let $\delta_s$ be the s-restricted isometry constant for a matrix $\Phi \in \mathbb{H}^{m\times n}$ for some $s \in \{1,\dots,n\}$. For any pair of $\mathbf{x},\mathbf{y} \in \mathbb{H}^n$ with disjoint supports and such that $\|\mathbf{x}\|_0 \leq s_1$ and $\|\mathbf{y}\|_0 \leq s_2$, where $s_1 + s_2 \leq n$,*

$$|\langle \Phi\mathbf{x}, \Phi\mathbf{y}\rangle| \leq \delta_{s_1+s_2} \|\mathbf{x}\|_2 \|\mathbf{y}\|_2.$$

*Proof.* In this proof, we will use the following notation: for any $\mathbf{x} \in \mathbb{H}^n$ and a set of indices $S \subset \{1,\dots,n\}$ with $\#S = s$, let $\mathbf{x}_{|S} = (x_{i_1},\dots,x_{i_s})$ for $S = \{i_1,\dots,i_s\}$.

Take any vectors $\mathbf{x},\mathbf{y} \in \mathbb{H}^n$ that satisfy the assumptions and denote $S = \operatorname{supp}(\mathbf{x}) \cup \operatorname{supp}(\mathbf{y})$. Obviously, $\#S = s_1 + s_2$. Since $\mathbf{x}$ and $\mathbf{y}$ have disjoint supports, they are orthogonal, which means that

$$\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}_{|S}, \mathbf{y}_{|S}\rangle = 0.$$

Using the Cauchy-Schwarz inequality and Lemma 3, we get

$$\begin{aligned}
|\langle \Phi\mathbf{x}, \Phi\mathbf{y}\rangle| &= \left|\langle \Phi_S\mathbf{x}_{|S}, \Phi_S\mathbf{y}_{|S}\rangle - \langle \mathbf{x}_{|S}, \mathbf{y}_{|S}\rangle\right| = \left|\langle (\Phi_S^*\Phi_S - \mathbf{Id})\,\mathbf{x}_{|S}, \mathbf{y}_{|S}\rangle\right| \\
&\leq \left\|(\Phi_S^*\Phi_S - \mathbf{Id})\,\mathbf{x}_{|S}\right\|_2 \left\|\mathbf{y}_{|S}\right\|_2 \leq \|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to 2} \left\|\mathbf{x}_{|S}\right\|_2 \left\|\mathbf{y}_{|S}\right\|_2 \\
&\leq \delta_{s_1+s_2} \left\|\mathbf{x}_{|S}\right\|_2 \left\|\mathbf{y}_{|S}\right\|_2,
\end{aligned}$$

which finishes the proof, since $\left\|\mathbf{x}_{|S}\right\|_2 = \|\mathbf{x}\|_2$ and $\left\|\mathbf{y}_{|S}\right\|_2 = \|\mathbf{y}\|_2$. □

# 4. STABLE RECONSTRUCTION FROM NOISY DATA

As we mentioned in the introduction, our aim is to reconstruct a quaternion signal from a limited number of its linear measurements with quaternion coefficients. We will also assume the presence of white noise with bounded $\ell_2$ quaternion norm. The observables are, therefore, given by

$$\mathbf{y} = \Phi\mathbf{x} + \mathbf{e}, \quad \text{where} \quad \mathbf{x} \in \mathbb{H}^n,\ \Phi \in \mathbb{H}^{m\times n},\ \mathbf{y} \in \mathbb{H}^m \text{ and } \mathbf{e} \in \mathbb{H}^m \text{ with } \|\mathbf{e}\|_2 \leq \eta$$

for some $m \leq n$ and $\eta \geq 0$.

We will use the following notation: for any $\mathbf{h} \in \mathbb{H}^n$ and a set of indices $T \subset \{1,\ldots,n\}$, the vector $\mathbf{h}_T \in \mathbb{H}^n$ is supported on $T$ with the following entries

$$(\mathbf{h}_T)_i = \begin{cases} h_i & \text{if } i \in T, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where} \quad \mathbf{h} = (h_1,\ldots,h_n)^T.$$

The complement of $T \subset \{1,\ldots,n\}$ will be denoted by $T^c = \{1,\ldots,n\} \setminus T$ and the symbol $\mathbf{x}_s$ will be used for the best $s$-sparse approximation of the vector $\mathbf{x}$, i.e. an $s$-sparse vector whose $s$ entries coincide with the coordinates of $\mathbf{x}$ with the biggest norms and other equal zero [13].

The following result is a generalization of [6, Theorem 1.3] and [1, Theorem 4.1] to the full quaternion case. It also improves the error estimate's constants from [1, Theorem 4.1].

**Theorem 5.** *Let a quaternion matrix $\Phi \in \mathbb{H}^{m\times n}$ satisfy the $2s$-restricted isometry property with a constant $\delta_{2s} < \sqrt{2}-1$ and let $\eta \geq 0$. Then, for any $\mathbf{x} \in \mathbb{H}^n$ and $\mathbf{e} \in \mathbb{H}^m$ with $\|\mathbf{e}\|_2 \leq \eta$, if $\mathbf{y} = \Phi\mathbf{x} + \mathbf{e}$, the solution $\mathbf{x}^\#$ of the problem*

$$\operatorname*{arg\,min}_{\mathbf{z}\in\mathbb{H}^n} \|\mathbf{z}\|_1 \quad \textit{subject to} \quad \|\Phi\mathbf{z} - \mathbf{y}\|_2 \leq \eta \tag{2}$$

*satisfies*

$$\left\|\mathbf{x}^\# - \mathbf{x}\right\|_2 \leq \frac{C_0}{\sqrt{s}} \|\mathbf{x} - \mathbf{x}_s\|_1 + C_1\eta \tag{3}$$

*with constants*

$$C_0 = 2 \cdot \frac{1 + \left(\sqrt{2}-1\right)\delta_{2s}}{1 - \left(\sqrt{2}+1\right)\delta_{2s}}, \quad C_1 = \frac{4\sqrt{1+\delta_{2s}}}{1 - \left(\sqrt{2}+1\right)\delta_{2s}},$$

*where $\mathbf{x}_s$ denotes the best $s$-sparse approximation of $\mathbf{x}$.*

*Proof.* Denote

$$\mathbf{h} = \mathbf{x}^\# - \mathbf{x}$$

and decompose $\mathbf{h}$ into a sum of vectors $\mathbf{h}_{T_0}, \mathbf{h}_{T_1}, \mathbf{h}_{T_2} \ldots$ in the following way: let $T_0$ be the set of $s$ indices of $\mathbf{x}$ coordinates with largest quaternion norms (hence $\mathbf{x}_s = \mathbf{x}_{T_0}$); $T_1$ be the set of indices of $\mathbf{h}_{T_0^c}$ coordinates with largest norms, $T_2$ be the set of $s$ indices of $\mathbf{h}_{(T_0\cup T_1)^c}$ coordinates with largest norms, etc. Then, obviously all $\mathbf{h}_{T_j}$ are $s$-sparse and have disjoint supports.

To prove (3), one can use the obvious inequality

$$\|\mathbf{h}\|_2 \leq \|\mathbf{h}_{T_0\cup T_1}\|_2 + \left\|\mathbf{h}_{(T_0\cup T_1)^c}\right\|_2$$

and show that the following two estimations hold

$$\left\|\mathbf{h}_{(T_0\cup T_1)^c}\right\|_2 \leq \|\mathbf{h}_{T_0}\|_2 + 2e \leq \|\mathbf{h}_{T_0\cup T_1}\|_2 + 2e, \tag{4}$$

$$\|\mathbf{h}_{T_0\cup T_1}\|_2 \leq \frac{1}{1-\beta}(\alpha\cdot\eta + 2\beta\cdot e), \tag{5}$$

where $e = \frac{1}{\sqrt{s}}\|\mathbf{x}-\mathbf{x}_s\|_1$ and

$$\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \quad \beta = \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}},$$

as long as $\beta < 1$, which is true for $\delta_{2s} < \sqrt{2}-1$. Taking $C_0 = \frac{4\beta}{1-\beta}+2$ and $C_1 = \frac{2\alpha}{1-\beta}$, we get the conclusion of the theorem.

We omit the proof of the estimate (4), since it is the same as its counterpart in the proof of [1, Theorem 4.1]. The proof of the inequality (5) is again very similar to the analogous part of the proof of [1, Theorem 4.1] – one should carefully follow its course using Lemma 4 from Section 3 instead of [1, Lemma 3.3], and thus obtaining

$$\left|\left\langle \Phi\mathbf{h}_{T_i}, \Phi\mathbf{h}_{T_j}\right\rangle\right| \leq \delta_{2s}\cdot\|\mathbf{h}_{T_i}\|_2\cdot\left\|\mathbf{h}_{T_j}\right\|_2 \quad \text{for} \quad i=0,1 \quad \text{and} \quad j\geq 2.$$

The remaining steps remain unchanged leading directly to (5), which concludes the proof. $\square$

# 5. STABLE RECONSTRUCTION FROM EXACT DATA

Let us now assume that our observables are exact, i.e.

$$\mathbf{y} = \Phi\mathbf{x}, \quad \text{where} \quad \mathbf{x}\in\mathbb{H}^n,\ \Phi\in\mathbb{H}^{m\times n},\ \mathbf{y}\in\mathbb{H}^m.$$

The below-mentioned result is a natural corollary of Theorem 5 for $\eta = 0$.

**Corollary 6.** *Let $\Phi\in\mathbb{H}^{m\times n}$ satisfy the $2s$-restricted isometry property with a constant $\delta_{2s} < \sqrt{2}-1$. Then for any $\mathbf{x}\in\mathbb{H}^n$ and $\mathbf{y}=\Phi\mathbf{x}\in\mathbb{H}^m$, the solution $\mathbf{x}^\#$ of the problem*

$$\arg\min_{\mathbf{z}\in\mathbb{H}^n}\|\mathbf{z}\|_1 \quad \textit{subject to} \quad \Phi\mathbf{z}=\mathbf{y} \tag{6}$$

*satisfies*

$$\left\|\mathbf{x}^{\#} - \mathbf{x}\right\|_1 \leq C_0 \left\|\mathbf{x} - \mathbf{x}_s\right\|_1 \tag{7}$$

*and*

$$\left\|\mathbf{x}^{\#} - \mathbf{x}\right\|_2 \leq \frac{C_0}{\sqrt{s}} \left\|\mathbf{x} - \mathbf{x}_s\right\|_1 \tag{8}$$

*with constant $C_0$ as in the Theorem 5. In particular, if* $\mathbf{x}$ *is s-sparse and there is no noise, then the reconstruction by $\ell_1$-norm minimization is exact.*

We skip the proof since it is identical to that of [1, Corollary 5.1]. We encourage the reader to follow the reasoning presented therein.

We conjecture that the requirement $\delta_{2s} < \sqrt{2} - 1$ is not optimal – there are known refinements of this condition for real signals (see e.g. [13, Chapter 6] for references). On the other hand, the authors of [5] constructed examples of $s$-sparse real signals which can not be uniquely reconstructed via $\ell_1$-norm minimization for $\delta_s > \frac{1}{3}$. This gives an obvious upper bound for $\delta_s$ also for the general quaternion case.

# 6. QUATERNION GAUSSIAN RANDOM MATRICES

For a real random variable $X$, we will denote its expectation (mean) by $\mathbb{E}X$ and its variance by $\operatorname{Var}X$. For Gamma distribution $\Gamma(\alpha, \beta)$ with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$, i.e., random variable $X$ with the probability density function

$$\gamma_{\alpha,\beta}(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{for } x \in (0, +\infty),$$

we have that

$$\mathbb{E}X = \frac{\alpha}{\beta} \quad \text{and} \quad \operatorname{Var}X = \frac{\alpha}{\beta^2}.$$

Recall that a sum of squares of $k$ independent standard Gaussian random variables $\mathcal{N}(0,1)$ has Chi-square distribution with $k$ degrees of freedom and $\chi^2(k) = \Gamma\left(\frac{k}{2}, \frac{1}{2}\right)$.

Quaternion random variables have not been studied as thoroughly as their real or complex counterparts so far. However, during the last two decades they attracted the attention of researchers both in theoretical and applied sciences [3, 27]. Quaternion random variable $X$ is defined by four real random variables

$$X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k}$$

and as such can be associated with the four-dimensional real random vector $(X_1, X_2, X_3, X_4)$. There are several definitions of a quaternion Gaussian random variable [27]. The most general (so-called *R-Gaussian*) calls the quaternion variable $X$ Gaussian if $(X_1, X_2, X_3, X_4)$ is a Gaussian random vector in $\mathbb{R}^4$.

Here we only consider *quaternion R-Gaussian random variables with independent components*. More precisely, we assume that

$$X_i \sim \mathcal{N}\left(0, \frac{\sigma^2}{4}\right),\ i = 1, \ldots, 4, \quad \text{and} \quad X_i \text{ are independent.}$$

Such quaternion variables $X = X_0 + X_1\mathbf{i} + X_2\mathbf{j} + X_3\mathbf{k}$ will be called Gaussian with mean zero and variance $\sigma^2$ and denoted by $X \sim \mathcal{N}_{\mathbb{H}}\left(0, \sigma^2\right)$.

In what follows, we consider quaternion random matrices with independent entries sampled from quaternion Gaussian distribution, which has been defined above. Let us emphasize once again that we always assume *independence of components* of quaternion Gaussian random variables.

**Lemma 7.** *Let* $\Phi = \left(\phi_{ij}\right)$ *be an* $m \times n$ *quaternion Gaussian random matrix whose entries are independent random variables with the distribution* $\mathcal{N}_{\mathbb{H}}\left(0, \frac{1}{m}\right)$ *and let* $\mathbf{0} \neq \mathbf{x} \in \mathbb{H}^n$. *Then the random variable*

$$\mathcal{R} = \frac{\|\Phi\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2}$$

*has Gamma distribution* $\Gamma(2m, 2m)$ *and it does not depend on* $\mathbf{x}$. *In particular,* $\mathbb{E}\mathcal{R} = 1$ *and* $\operatorname{Var}\mathcal{R} = \frac{1}{2m}$.

*Proof.* Since $\frac{\Phi\mathbf{x}}{\|\mathbf{x}\|_2} = \Phi\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2^2}\right)$, without loss of generality we can assume that $\|\mathbf{x}\|_2 = 1$ and hence, $\mathcal{R} = \|\Phi\mathbf{x}\|_2^2$. Let us decompose the matrix $\Phi$ into its components:

$$\Phi = \Phi_{\mathbf{r}} + \Phi_{\mathbf{i}}\mathbf{i} + \Phi_{\mathbf{j}}\mathbf{j} + \Phi_{\mathbf{k}}\mathbf{k}, \quad \text{where} \quad \Phi_{\mathbf{r}}, \Phi_{\mathbf{i}}, \Phi_{\mathbf{j}}, \Phi_{\mathbf{k}} \text{ are real Gaussian random matrices,}$$

and analogously every matrix entry can be written as

$$\phi_{ij} = \phi_{\mathbf{r},ij} + \phi_{\mathbf{i},ij}\mathbf{i} + \phi_{\mathbf{j},ij}\mathbf{j} + \phi_{\mathbf{k},ij}\mathbf{k} \quad \text{with} \quad \phi_{e,ij} \sim \mathcal{N}\left(0, \frac{1}{m}\right) \quad \text{for} \quad e \in \{\mathbf{r}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

In the same way we denote components of the vector $\mathbf{x} = (x_1, \ldots, x_n)^T$.

Let $\Phi\mathbf{x} = \mathbf{y} = (y_1, \ldots, y_m)^T$, then

$$\begin{aligned}
y_k &= \sum_{\ell=1}^{n} \phi_{k\ell} x_\ell \\
&= \sum_{\ell=1}^{n} \left(\phi_{\mathbf{r},k\ell} + \phi_{\mathbf{i},k\ell}\mathbf{i} + \phi_{\mathbf{j},k\ell}\mathbf{j} + \phi_{\mathbf{k},k\ell}\mathbf{k}\right) \cdot \left(x_{\mathbf{r},\ell} + x_{\mathbf{i},\ell}\mathbf{i} + x_{\mathbf{j},\ell}\mathbf{j} + x_{\mathbf{k},\ell}\mathbf{k}\right) \\
&= \sum_{\ell=1}^{n} \left(\phi_{\mathbf{r},k\ell} x_{\mathbf{r},\ell} - \phi_{\mathbf{i},k\ell} x_{\mathbf{i},\ell} - \phi_{\mathbf{j},k\ell} x_{\mathbf{j},\ell} - \phi_{\mathbf{k},k\ell} x_{\mathbf{k},\ell}\right) \\
&\quad + \sum_{\ell=1}^{n} \left(\phi_{\mathbf{i},k\ell} x_{\mathbf{r},\ell} + \phi_{\mathbf{r},k\ell} x_{\mathbf{i},\ell} - \phi_{\mathbf{k},k\ell} x_{\mathbf{j},\ell} + \phi_{\mathbf{j},k\ell} x_{\mathbf{k},\ell}\right)\mathbf{i}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=1}^{n}(\phi_{\mathbf{j},k\ell}x_{\mathbf{r},\ell}+\phi_{\mathbf{k},k\ell}x_{\mathbf{i},\ell}+\phi_{\mathbf{r},k\ell}x_{\mathbf{j},\ell}-\phi_{\mathbf{i},k\ell}x_{\mathbf{k},\ell})\mathbf{j}\\
&+\sum_{\ell=1}^{n}(\phi_{\mathbf{k},k\ell}x_{\mathbf{r},\ell}-\phi_{\mathbf{j},k\ell}x_{\mathbf{i},\ell}+\phi_{\mathbf{i},k\ell}x_{\mathbf{j},\ell}+\phi_{\mathbf{r},k\ell}x_{\mathbf{k},\ell})\mathbf{k}\\
=:\ & y_{\mathbf{r},k}+y_{\mathbf{i},k}\mathbf{i}+y_{\mathbf{j},k}\mathbf{j}+y_{\mathbf{k},k}\mathbf{k}.
\end{aligned}
$$

Recall that

$$
\phi_{e,ij}\sim\mathcal{N}\left(0,\frac{1}{4m}\right)\quad\text{for}\quad e\in\{\mathbf{r},\mathbf{i},\mathbf{j},\mathbf{k}\},\quad i\in\{1,\dots,m\},\quad j\in\{1,\dots,n\}.
$$

Hence, for each $e\in\{\mathbf{r},\mathbf{i},\mathbf{j},\mathbf{k}\}$ and every $k\in\{1,\dots,m\}$, random variables $y_{e,k}$ are independent Gaussian variables (as linear combinations of Gaussian random variables) with $\mathbb{E}\,y_{e,k}=0$ and $\operatorname{Var}y_{e,k}=\frac{1}{4m}$ – since all $\phi_{e,k\ell}$ are independent and

$$
\begin{aligned}
\operatorname{Var}y_{\mathbf{r},k}&=\sum_{\ell=1}^{n}(x_{\mathbf{r},\ell}^2\operatorname{Var}\phi_{\mathbf{r},k\ell}+x_{\mathbf{i},\ell}^2\operatorname{Var}\phi_{\mathbf{i},k\ell}+x_{\mathbf{j},\ell}^2\operatorname{Var}\phi_{\mathbf{j},k\ell}+x_{\mathbf{k},\ell}^2\operatorname{Var}\phi_{\mathbf{k},k\ell})\\
&=\frac{1}{4m}\sum_{\ell=1}^{n}(x_{\mathbf{r},\ell}^2+x_{\mathbf{i},\ell}^2+x_{\mathbf{j},\ell}^2+x_{\mathbf{k},\ell}^2)=\frac{\|\mathbf{x}\|_2^2}{4m}=\frac{1}{4m}.
\end{aligned}
$$

For the remaining components we proceed analogously.

Independence of the variables $\phi_{e,k\ell}$ implies also independence of $y_{e,k}$ and $y_{e,\ell}$ for every fixed $e\in\{\mathbf{r},\mathbf{i},\mathbf{j},\mathbf{k}\}$ and for all pairs $k,\ell\in\{1,\dots,m\}$, $k\neq\ell$. In order to verify independence of $y_{\mathbf{r},k}$ and $y_{\mathbf{i},k}$, $k\in\{1,\dots,m\}$, observe that

$$
\begin{aligned}
\operatorname{Cov}(y_{\mathbf{r},k},y_{\mathbf{i},k})&=\mathbb{E}\,(y_{\mathbf{r},k}\cdot y_{\mathbf{i},k})-\mathbb{E}\,y_{\mathbf{r},k}\cdot\mathbb{E}\,y_{\mathbf{i},k}\\
&=\mathbb{E}\left[\left(\sum_{\ell=1}^{n}(\phi_{\mathbf{r},k\ell}x_{\mathbf{r},\ell}-\phi_{\mathbf{i},k\ell}x_{\mathbf{i},\ell}-\phi_{\mathbf{j},k\ell}x_{\mathbf{j},\ell}-\phi_{\mathbf{k},k\ell}x_{\mathbf{k},\ell})\right)\right.\\
&\qquad\left.\cdot\left(\sum_{p=1}^{n}(\phi_{\mathbf{i},kp}x_{\mathbf{r},p}+\phi_{\mathbf{r},kp}x_{\mathbf{i},p}-\phi_{\mathbf{k},kp}x_{\mathbf{j},p}+\phi_{\mathbf{j},kp}x_{\mathbf{k},p})\right)\right]\\
&=\sum_{\ell=1}^{n}\mathbb{E}\,\big((\phi_{\mathbf{r},k\ell}x_{\mathbf{r},\ell}-\phi_{\mathbf{i},k\ell}x_{\mathbf{i},\ell}-\phi_{\mathbf{j},k\ell}x_{\mathbf{j},\ell}-\phi_{\mathbf{k},k\ell}x_{\mathbf{k},\ell})\\
&\qquad\cdot(\phi_{\mathbf{i},k\ell}x_{\mathbf{r},\ell}+\phi_{\mathbf{r},k\ell}x_{\mathbf{i},\ell}-\phi_{\mathbf{k},k\ell}x_{\mathbf{j},\ell}+\phi_{\mathbf{j},k\ell}x_{\mathbf{k},\ell})\big)\\
&=\sum_{\ell=1}^{n}(x_{\mathbf{r},\ell}x_{\mathbf{i},\ell}\mathbb{E}\,\phi_{\mathbf{r},k\ell}^2-x_{\mathbf{i},\ell}x_{\mathbf{r},\ell}\mathbb{E}\,\phi_{\mathbf{i},k\ell}^2-x_{\mathbf{j},\ell}x_{\mathbf{k},\ell}\mathbb{E}\,\phi_{\mathbf{j},k\ell}^2+x_{\mathbf{k},\ell}x_{\mathbf{j},\ell}\mathbb{E}\,\phi_{\mathbf{k},k\ell}^2)\\
&=\frac{1}{4m}\sum_{\ell=1}^{n}(x_{\mathbf{r},\ell}x_{\mathbf{i},\ell}-x_{\mathbf{i},\ell}x_{\mathbf{r},\ell}-x_{\mathbf{j},\ell}x_{\mathbf{k},\ell}+x_{\mathbf{k},\ell}x_{\mathbf{j},\ell})=0,
\end{aligned}
$$

since $\mathbb{E}\,\phi_{e,k\ell}^2=\operatorname{Var}\phi_{e,k\ell}=\frac{1}{4m}$. In the same way one argues that covariance of the remaining pairs is also zero. Recall that real Gaussian random vectors have independent components if and only if their covariance equals zero.

Therefore, we get that all $y_{e,k}$, $e \in \{\mathbf{r}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $k \in \{1, \ldots, m\}$, are independent (real) random variables with distribution $\mathcal{N}\left(0, \frac{1}{4m}\right)$ and, hence, $\sqrt{4m}\, y_{e,k} \sim \mathcal{N}(0,1)$. This implies that

$$4m\mathcal{R} = \sum_{\ell=1}^{m} \left( \left(\sqrt{4m} y_{\mathbf{r},\ell}\right)^2 + \left(\sqrt{4m} y_{\mathbf{i},\ell}\right)^2 + \left(\sqrt{4m} y_{\mathbf{j},\ell}\right)^2 + \left(\sqrt{4m} y_{\mathbf{k},\ell}\right)^2 \right)$$

is a sum of $4m$ squares of independent standard Gaussian random variables and, consequently, $4m\mathcal{R}$ has Chi-square distribution $\chi^2(4m) = \Gamma(2m, \frac{1}{2})$. We conclude that $\mathcal{R}$ has distribution $\Gamma(2m, 2m)$, independently of $\mathbf{x}$. This random variable has mean $\frac{2m}{2m} = 1$ and variance $\frac{2m}{(2m)^2} = \frac{1}{2m}$. □

As we previously remarked, in the real case $\mathcal{R}$ has distribution $\Gamma\left(\frac{m}{2}, \frac{m}{2}\right)$, that is with four times bigger variance [18]. It explains the aforementioned better results of quaternion sparse vectors reconstruction compared with the real case (see Fig. 1 and Fig. 2(a) in Section 8), since a quaternion Gaussian random matrix statistically has smaller restricted isometry constant than its real counterpart.

Let us now proceed with the tools needed for the proof of the main result of Section 7. Recall that a real random variable $X$ is called *sub-exponential* (*locally sub-Gaussian*) [28] if there exist $\sigma^2 > 0$ and $\delta > 0$ such that

$$\mathbb{E}\left(\mathrm{e}^{t(X - \mathbb{E}X)}\right) \leq \exp\left(\frac{\sigma^2 t^2}{2}\right) \quad \text{for } |t| \leq \delta.$$

We will denote it $X \sim \mathrm{SubExp}(\sigma^2, \delta)$. Equivalently, one may write that

$$M(t) = \mathbb{E}\left(\mathrm{e}^{tX}\right) \leq \exp\left(t\,\mathbb{E}X + \frac{\sigma^2 t^2}{2}\right) \quad \text{for } |t| \leq \delta,$$

where $M(t)$ is the moment generating function.

It is known that a random variable with Gamma distribution is sub-exponential with any $\sigma^2 > \mathrm{Var} X$ for some $\delta > 0$ [28]. Below, we recall a simple proof of this fact, with $\sigma^2$ and $\delta$ chosen for our purposes, in which we shall use the following form of the moment generating function of the Gamma distribution $\Gamma(\alpha, \beta)$:

$$M(t) = \frac{1}{\left(1 - \frac{t}{\beta}\right)^{\alpha}} \quad \text{for } t < \beta.$$

**Lemma 8.** *Let $X$ have distribution $\Gamma(\alpha, \beta)$. Then $X \sim \mathrm{SubExp}\left(\frac{5}{2}\mathrm{Var} X, \frac{\beta}{5}\right)$.*

*Proof.* Indeed, take $|t| \leq \frac{\beta}{5}$, which means that $\frac{|t|}{\beta} \leq \frac{1}{5}$. Then

$$\mathbb{E}\left(e^{t(X - \frac{\alpha}{\beta})}\right) = \frac{1}{\left(1 - \frac{t}{\beta}\right)^{\alpha}} \cdot \mathrm{e}^{-t\frac{\alpha}{\beta}} = \left(1 + \frac{t}{\beta} + \frac{(\frac{t}{\beta})^2}{1 - \frac{t}{\beta}}\right)^{\alpha} \cdot \mathrm{e}^{-t\frac{\alpha}{\beta}}$$

$$
\begin{aligned}
&\overset{1-\frac{t}{\beta}\geq\frac{4}{5}}{\leq} \left(1+\frac{t}{\beta}+\frac{5}{4}\left(\frac{t}{\beta}\right)^2\right)^{\alpha}\cdot \mathrm{e}^{-t\frac{\alpha}{\beta}}\\
&\leq \exp\left(\alpha\cdot\left(\frac{t}{\beta}+\frac{5}{4}\left(\frac{t}{\beta}\right)^2\right)\right)\cdot\exp\left(-t\frac{\alpha}{\beta}\right)\\
&= \exp\left(\frac{1}{2}\cdot\frac{5}{2}\frac{\alpha}{\beta^2}\cdot t^2\right),
\end{aligned}
$$

where we used well known estimate $1+x\leq \mathrm{e}^x$ for $x\in\mathbb{R}$. Hence, $X$ is indeed sub-exponential $\mathrm{SubExp}(\sigma^2,\delta)$ with parameters $\sigma^2=\frac{5}{2}\frac{\alpha}{\beta^2}=\frac{5}{2}\operatorname{Var}X$ and $\delta=\frac{\beta}{5}$. □

We will use the following known fact [28]: if $X\sim\mathrm{SubExp}(\sigma^2,\delta)$, then

$$
\mathbb{P}\left(|X-\mathbb{E}X|\geq t\right)\leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right)\quad\text{for}\quad 0\leq t\leq\sigma^2\delta.
$$

**Corollary 9.** *The random variable $\mathcal{R}\sim\Gamma(2m,2m)$ from Lemma 7 is sub-exponential with parameters $\sigma^2=\frac{5}{2}\cdot\frac{1}{2m}=\frac{5}{4m}$ and $\delta=\frac{2m}{5}$. Hence,*

$$
\mathbb{P}\left(|\mathcal{R}-1|\geq t\right)\leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right)\quad\textit{for}\quad 0\leq t\leq\frac{1}{2},
$$

*and therefore*

$$
\forall_{\mathbf{0}\neq\mathbf{x}\in\mathbb{H}^n}\quad\forall_{0\leq t\leq\frac{1}{2}}\quad \mathbb{P}\left(\left|\|\Phi\mathbf{x}\|_2^2-\|\mathbf{x}\|_2^2\right|\geq t\,\|\mathbf{x}\|_2^2\right)\leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right). \tag{9}
$$

# 7. QUATERNION GAUSSIAN MATRICES SATISFY THE RIP

As it was stated in Section 3, we say that a deterministic matrix $\Phi\in\mathbb{H}^{m\times n}$ satisfies the $s$-restricted isometry property (for quaternion vectors) with a constant $\delta_s\geq 0$ if the inequalities (1) hold for all $s$-sparse vectors $\mathbf{x}\in\mathbb{H}^n$. The smallest number $\delta_s\geq 0$ with this property is called the $s$-restricted isometry constant. Without loss of generality, one can only consider quaternion $s$-sparse unit vectors, i.e. $\|\mathbf{x}\|_2=1$. Moreover, in Lemma 3 we proved that

$$
\delta_s=\max_{S\subset\{1,\ldots,n\},\#S\leq s}\|\Phi_S^*\Phi_S-\mathbf{Id}\|_{2\to 2}, \tag{10}
$$

where $\delta_s$ is the $s$-restricted isometry constant of $\Phi\in\mathbb{H}^{m\times n}$ and $\Phi_S$ is the submatrix of $\Phi$ consisting of columns with indices from $S\subset\{1,\ldots,n\}$.

In this chapter we consider random matrix $\Phi$ and $s$-restricted isometry constants of its realizations. Since it does not lead to confusion, in what follows we will use the same symbol $\delta_s$ for the random variable (defined on the same probability space as $\Phi$) giving the $s$-restricted isometry contstant of each realization of $\Phi$.

We begin with the following result in which we fix the support set.

**Lemma 10.** *Let $\Phi \in \mathbb{H}^{m\times n}$ be an $m\times n$ quaternion Gaussian matrix whose entries $\phi_{ij}$ are independent quaternion random variables with distribution $\mathcal{N}_{\mathbb{H}}(0,\frac{1}{m})$. Moreover, let the set $S\subset\{1,\dots,n\}$ be such that $\#S = s\leq n$. For any $\delta\in\left(0,\frac{1}{\sqrt{3}}\right)$ and $\varepsilon\in(0,1)$, if*

$$m \geq \frac{10}{3}\delta^{-2}\left(14s+\ln\left(\frac{2}{\varepsilon}\right)\right),$$

*then, with probability at least $1-\varepsilon$,*

$$\|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to 2} < \delta.$$

*Proof.* Fix a set $S\subset\{1,\dots,n\}$ with $\#S = s$ and denote

$$\mathcal{A}_S = \{\mathbf{x}\in\mathbb{H}^n : \quad \operatorname{supp}\mathbf{x}\subset S \quad \wedge \quad \|\mathbf{x}\|_2 = 1\}.$$

This set can be associated with the unit sphere $\mathcal{S}^{4s-1}$ in $\mathbb{R}^{4s}$.

Take a number $0<\gamma<\frac{1}{2}$ (the exact value of $\gamma$ will be specified later). By [13, Proposition C.3], there exists a $\gamma$-covering $\mathcal{A}_\gamma$ of $\mathcal{A}_S$ such that

$$\#\mathcal{A}_\gamma \leq \left(1+\frac{2}{\gamma}\right)^{4s}.$$

For any $0\leq\tilde{\delta}\leq\frac{1}{2}$, using (9) from Corollary 9, we get

$$\begin{aligned}\mathbb{P}\left(\exists_{\mathbf{y}\in\mathcal{A}_\gamma}\left|\|\Phi\mathbf{y}\|_2^2-\|\mathbf{y}\|_2^2\right|\geq\tilde{\delta}\,\|\mathbf{y}\|_2^2\right) &= \mathbb{P}\left(\bigcup_{\mathbf{y}\in\mathcal{A}_\gamma}\left\{\left|\|\Phi\mathbf{y}\|_2^2-\|\mathbf{y}\|_2^2\right|\geq\tilde{\delta}\,\|\mathbf{y}\|_2^2\right\}\right)\\ &\leq \sum_{\mathbf{y}\in\mathcal{A}_\gamma}\mathbb{P}\left(\left|\|\Phi\mathbf{y}\|_2^2-\|\mathbf{y}\|_2^2\right|\geq\tilde{\delta}\,\|\mathbf{y}\|_2^2\right)\\ &\leq \#\mathcal{A}_\gamma\cdot 2\exp\left(-\frac{\tilde{\delta}^2}{2\sigma^2}\right)\\ &\leq 2\left(1+\frac{2}{\gamma}\right)^{4s}\exp\left(-\frac{\tilde{\delta}^2}{2\sigma^2}\right)\end{aligned}$$

with $\sigma^2=\frac{5}{4m}$. This implies that

$$\mathbb{P}\left(\forall_{\mathbf{y}\in\mathcal{A}_\gamma}\quad\left|\|\Phi\mathbf{y}\|_2^2-\|\mathbf{y}\|_2^2\right|<\tilde{\delta}\,\|\mathbf{y}\|_2^2\right)\geq 1-2\left(1+\frac{2}{\gamma}\right)^{4s}\exp\left(-\frac{\tilde{\delta}^2}{2\sigma^2}\right).$$

Since $\mathcal{A}_\gamma\subset\mathcal{A}_S$, we obviously have

$$\forall_{\mathbf{y}\in\mathcal{A}_\gamma}\quad\left|\|\Phi\mathbf{y}\|_2^2-\|\mathbf{y}\|_2^2\right|<\tilde{\delta}\,\|\mathbf{y}\|_2^2\quad\Leftrightarrow\quad\forall_{\mathbf{y}\in\mathcal{A}_\gamma}\quad\left|\|\Phi_S\mathbf{y}_S\|_2^2-\|\mathbf{y}_S\|_2^2\right|<\tilde{\delta}\,\|\mathbf{y}_S\|_2^2 \tag{11}$$

and
$$\left| \|\Phi_S \mathbf{y}_S\|_2^2 - \|\mathbf{y}_S\|_2^2 \right| = |\langle (\Phi_S^* \Phi_S - \mathbf{Id})\mathbf{y}_S, \mathbf{y}_S \rangle| \,.$$

For a matrix $\Phi$ satisfying the left (or equivalently the right) side of (11), denote $\Psi = \Phi_S^*\Phi_S - \mathbf{Id}$. Since all vectors in $\mathcal{A}_\gamma$ are unit vectors supported on $S$, we have

$$\forall_{\mathbf{y}\in\mathcal{A}_\gamma} \quad |\langle \Psi \mathbf{y}_S, \mathbf{y}_S \rangle| < \tilde{\delta}\, \|\mathbf{y}_S\|_2^2 = \tilde{\delta}\, \|\mathbf{y}\|_2^2 = \tilde{\delta}.$$

By the definition of a $\gamma$-covering, for every $\mathbf{x} \in \mathcal{A}_S$ there is some $\mathbf{y} \in \mathcal{A}_\gamma$ such that $\|\mathbf{x}-\mathbf{y}\|_2 \le \gamma < \frac{1}{2}$. Since both $\mathbf{x}$ and $\mathbf{y}$ are unit vectors supported on $S$, using properties of the Hermitian norm and quaternion Cauchy-Schwarz inequality we get

$$\begin{aligned} |\langle \Psi \mathbf{x}_S, \mathbf{x}_S \rangle| &= |\langle \Psi \mathbf{y}_S, \mathbf{y}_S \rangle + \langle \Psi \mathbf{x}_S, \mathbf{x}_S - \mathbf{y}_S \rangle + \langle \Psi(\mathbf{x}_S - \mathbf{y}_S), \mathbf{y}_S \rangle| \\ &\le |\langle \Psi \mathbf{y}_S, \mathbf{y}_S \rangle| + \|\Psi\|_{2\to2} \|\mathbf{x}\|_2 \|\mathbf{x}-\mathbf{y}\|_2 + \|\Psi\|_{2\to2} \|\mathbf{x}-\mathbf{y}\|_2 \|\mathbf{y}\|_2 \\ &< \tilde{\delta} + 2\gamma \|\Psi\|_{2\to2} \,. \end{aligned}$$

In view of Lemma 1, since $\Psi$ is Hermitian, taking supremum over all $\mathbf{x} \in \mathcal{A}_S$, we obtain

$$\|\Psi\|_{2\to2} < \tilde{\delta} + 2\gamma\|\Psi\|_{2\to2} \quad \Rightarrow \quad \|\Psi\|_{2\to2} < \frac{\tilde{\delta}}{1-2\gamma}.$$

Denoting $\delta = \frac{\tilde{\delta}}{1-2\gamma} \le \frac{1}{2} \cdot \frac{1}{1-2\gamma}$, we get

$$\begin{aligned} \mathbb{P}\left(\|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to2} < \delta\right) &\ge \mathbb{P}\left(\forall_{\mathbf{y}\in\mathcal{A}_\gamma} \quad \left| \|\Phi \mathbf{y}\|_2^2 - \|\mathbf{y}\|_2^2 \right| < \tilde{\delta}\, \|\mathbf{y}\|_2^2 \right) \\ &\ge 1 - 2\left(1+\frac{2}{\gamma}\right)^{4s} \exp\left(-\frac{\tilde{\delta}^2}{2\sigma^2}\right) \\ &= 1 - 2\left(1+\frac{2}{\gamma}\right)^{4s} \exp\left(-\frac{2}{5}\delta^2(1-2\gamma)^2 m\right). \end{aligned}$$

It implies that if

$$m \ge \frac{5}{2} \cdot \frac{\delta^{-2}}{(1-2\gamma)^2} \left(4s \cdot \ln\left(1+\frac{2}{\gamma}\right) + \ln\left(\frac{2}{\varepsilon}\right)\right),$$

then
$$\mathbb{P}\left(\|\Phi_S^*\Phi_S - \mathbf{Id}\|_{2\to2} < \delta\right) \ge 1 - \varepsilon. \tag{12}$$

Taking $\gamma = \frac{2}{e^{7/2}-1} \approx 6.23 \cdot 10^{-2}$, for which $\frac{1}{(1-2\gamma)^2} \le \frac{4}{3}$ and $\ln\left(1+\frac{2}{\gamma}\right) = \frac{7}{2}$, we finally obtain that for any positive $\delta \le \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$, if

$$m \ge \frac{10}{3}\delta^{-2}\left(14s + \ln\left(\frac{2}{\varepsilon}\right)\right),$$

then (12) holds, which concludes the proof. □

We are ready to prove the main result of this section.

**Theorem 11.** *Let $\Phi$ be an $m\times n$ quaternion Gaussian matrix whose entries $\phi_{ij}$ are independent quaternion random variables with distribution $\mathcal{N}_{\mathbb{H}}(0,\frac{1}{m})$. For any $\delta\in\left(0,\frac{1}{\sqrt{3}}\right)$ and $\varepsilon\in(0,1)$, if*

$$m\geq\frac{10}{3}\delta^{-2}\left(15s+\ln\left(\frac{2}{\varepsilon}\right)+s\ln\left(\frac{n}{s}\right)\right),$$

*then with probability at least $1-\varepsilon$ the s-restricted isometry constant $\delta_s$ of $\Phi$ satisfies $\delta_s<\delta$.*

*Proof.* Using (10), the proof of Lemma 10 and well known estimates of the Newton's symbol we get

$$\begin{aligned}\mathbb{P}(\delta_s\geq\delta)&=\mathbb{P}\left(\max_{S\colon \#S=s}\|\Phi_S^*\Phi_S-\mathbf{Id}\|_{2\to2}\geq\delta\right)\\&=\mathbb{P}\left(\exists_{S\colon \#S=s}\quad\|\Phi_S^*\Phi_S-\mathbf{Id}\|_{2\to2}\geq\delta\right)\\&=\mathbb{P}\left(\bigcup_{S\colon \#S=s}\left\{\|\Phi_S^*\Phi_S-\mathbf{Id}\|_{2\to2}\geq\delta\right\}\right)\\&\leq\sum_{S\colon \#S=s}\mathbb{P}\left(\|\Phi_S^*\Phi_S-\mathbf{Id}\|_{2\to2}\geq\delta\right)\\&\leq\binom{n}{s}\cdot2\left(1+\frac{2}{\gamma}\right)^{4s}\exp\left(-\frac{2}{5}\delta^2(1-2\gamma)^2m\right)\\&\leq2\left(\frac{en}{s}\right)^s\left(1+\frac{2}{\gamma}\right)^{4s}\exp\left(-\frac{2}{5}\delta^2(1-2\gamma)^2m\right).\end{aligned}$$

Therefore, if

$$m\geq\frac{5}{2}\cdot\frac{\delta^{-2}}{(1-2\gamma)^2}\left(s\ln\left(\frac{en}{s}\right)+4s\cdot\ln\left(1+\frac{2}{\gamma}\right)+\ln\left(\frac{2}{\varepsilon}\right)\right),$$

then $\mathbb{P}(\delta_s<\delta)\geq1-\varepsilon$. Taking again $\gamma=\frac{2}{e^{7/2}-1}$, we get the result. □

## 8. NUMERICAL EXPERIMENT

In [1], we presented the results of numerical experiments of sparse quaternion vector $\mathbf{x}$ reconstruction from its linear measurements $\mathbf{y}=\Phi\mathbf{x}$ in the case of real-valued measurement matrix $\Phi$. Those experiments were inspired by the articles [2, 17, 30] and involved expressing the quaternion $\ell_1$-norm minimization problem in terms of the second-order cone programming (SOCP).

In view of the main results of this paper (Theorem 5, Corollary 6), and having in mind that quaternion Gaussian random matrices satisfy (with overwhelming probability) the restricted isometry property (Theorem 11), we performed similar experiments in the case of quaternion matrix – as in [30]. As in Section 6, we consider an $m \times n$ quaternion Gaussian random matrix $\Phi = (\phi_{ij})$ whose entries are independent random variables with the distribution $\mathcal{N}_{\mathbb{H}}\left(0, \frac{1}{m}\right)$. In what follows, we consider only the case of noiseless measurements, i.e. we solve the problem (6).

Recall, after [30], that problem (6) is equivalent to

$$\arg\min_{t \in \mathbb{R}_+} t \quad \text{subject to} \quad \mathbf{y} = \Phi\mathbf{z},\ \|\mathbf{z}\|_1 \leq t. \tag{13}$$

We decompose vectors $\mathbf{y} \in \mathbb{H}^m$ and $\mathbf{z} \in \mathbb{H}^n$ into real vectors representing their real parts and components of their imaginary parts

$$\mathbf{y} = \mathbf{y_r} + \mathbf{y_i}\mathbf{i} + \mathbf{y_j}\mathbf{j} + \mathbf{y_k}\mathbf{k}, \qquad \mathbf{z} = \mathbf{z_r} + \mathbf{z_i}\mathbf{i} + \mathbf{z_j}\mathbf{j} + \mathbf{z_k}\mathbf{k},$$

where $\mathbf{y_r}, \mathbf{y_i}, \mathbf{y_j}, \mathbf{y_k} \in \mathbb{R}^m$, $\mathbf{z_r}, \mathbf{z_i}, \mathbf{z_j}, \mathbf{z_k} \in \mathbb{R}^n$. Denote

$$\mathbf{z_r} = (z_{\mathbf{r},1}, \ldots, z_{\mathbf{r},n})^T,\ \mathbf{z_i} = (z_{\mathbf{i},1}, \ldots, z_{\mathbf{i},n})^T,\ \mathbf{z_j} = (z_{\mathbf{j},1}, \ldots, z_{\mathbf{j},n})^T,\ \mathbf{z_k} = (z_{\mathbf{k},1}, \ldots, z_{\mathbf{k},n})^T,$$

and let $\phi_k \in \mathbb{H}^m$, $k \in \{1, \ldots, n\}$ be the $k$-th column of the matrix $\Phi$. Again, decompose as previously

$$\phi_k = \phi_{\mathbf{r},k} + \phi_{\mathbf{i},k}\mathbf{i} + \phi_{\mathbf{j},k}\mathbf{j} + \phi_{\mathbf{k},k}\mathbf{k},$$

where $\phi_{\mathbf{r},k}, \phi_{\mathbf{i},k}, \phi_{\mathbf{j},k}, \phi_{\mathbf{k},k} \in \mathbb{R}^m$. Note that the second constraint in (13) can be written in the form

$$\left\|(z_{\mathbf{r},k}, z_{\mathbf{i},k}, z_{\mathbf{j},k}, z_{\mathbf{k},k})^T\right\|_2 \leq t_k \quad \text{for } k \in \{1, \ldots, n\},$$

where $t_k$ are positive real numbers such that $\sum_{k=1}^{n} t_k = t$. Now we can rewrite (13) in the real-valued setup in the following way:

$$\begin{aligned} &\arg\min_{\tilde{\mathbf{z}} \in \mathbb{R}^n} \mathbf{c}^T\tilde{\mathbf{z}} \quad \text{subject to} \quad \tilde{\mathbf{y}} = \tilde{\Phi}\tilde{\mathbf{z}} \\ &\qquad\qquad \text{and} \quad \left\|(z_{\mathbf{r},k}, z_{\mathbf{i},k}, z_{\mathbf{j},k}, z_{\mathbf{k},k})^T\right\|_2 \leq t_k \quad \text{for } k \in \{1, \ldots, n\}, \end{aligned} \tag{14}$$

where

$$\tilde{\mathbf{z}} = (t_1, z_{\mathbf{r},1}, z_{\mathbf{i},1}, z_{\mathbf{j},1}, z_{\mathbf{k},1}, \ldots, t_n, z_{\mathbf{r},n}, z_{\mathbf{i},n}, z_{\mathbf{j},n}, z_{\mathbf{k},n})^T \in \mathbb{R}^{5n}, \tag{15}$$

$$\mathbf{c} = (1,0,0,0,0,\ldots,1,0,0,0,0)^T \in \mathbb{R}^{5n}, \tag{16}$$

$$\tilde{\mathbf{y}} = (\mathbf{y_r}^T, \mathbf{y_i}^T, \mathbf{y_j}^T, \mathbf{y_k}^T)^T \in \mathbb{R}^{4m}, \tag{17}$$

$$\tilde{\Phi} = \begin{pmatrix} \mathbf{0} & \phi_{\mathbf{r},1} & -\phi_{\mathbf{i},1} & -\phi_{\mathbf{j},1} & -\phi_{\mathbf{k},1} & \ldots & \mathbf{0} & \phi_{\mathbf{r},n} & -\phi_{\mathbf{i},n} & -\phi_{\mathbf{j},n} & -\phi_{\mathbf{k},n} \\ \mathbf{0} & \phi_{\mathbf{i},1} & \phi_{\mathbf{r},1} & -\phi_{\mathbf{k},1} & \phi_{\mathbf{j},1} & \ldots & \mathbf{0} & \phi_{\mathbf{i},n} & \phi_{\mathbf{r},n} & -\phi_{\mathbf{k},n} & \phi_{\mathbf{j},n} \\ \mathbf{0} & \phi_{\mathbf{j},1} & \phi_{\mathbf{k},1} & \phi_{\mathbf{r},1} & -\phi_{\mathbf{i},1} & \ldots & \mathbf{0} & \phi_{\mathbf{j},n} & \phi_{\mathbf{k},n} & \phi_{\mathbf{r},n} & -\phi_{\mathbf{i},n} \\ \mathbf{0} & \phi_{\mathbf{k},1} & -\phi_{\mathbf{j},1} & \phi_{\mathbf{i},1} & \phi_{\mathbf{r},1} & \ldots & \mathbf{0} & \phi_{\mathbf{k},n} & -\phi_{\mathbf{j},n} & \phi_{\mathbf{i},n} & \phi_{\mathbf{r},n} \end{pmatrix} \tag{18}$$

and $\tilde{\Phi} \in \mathbb{R}^{4m \times 5n}$.

This is a standard form of the SOCP, which can be solved using the SeDuMi toolbox for MATLAB [25]. The solution

$$\tilde{\mathbf{x}}^{\#} = \left(t_1, x^{\#}_{\mathbf{r},1}, x^{\#}_{\mathbf{i},1}, x^{\#}_{\mathbf{j},1}, x^{\#}_{\mathbf{k},1}, \ldots, t_n, x^{\#}_{\mathbf{r},n}, x^{\#}_{\mathbf{i},n}, x^{\#}_{\mathbf{j},n}, x^{\#}_{\mathbf{k},n}\right)^T \in \mathbb{R}^{5n} \tag{19}$$

to the problem (14) can easily be expressed as

$$\mathbf{x}^{\#} = \left(x^{\#}_{\mathbf{r},1} + x^{\#}_{\mathbf{i},1}\mathbf{i} + x^{\#}_{\mathbf{j},1}\mathbf{j} + x^{\#}_{\mathbf{k},1}\mathbf{k},\ \ldots,\ x^{\#}_{\mathbf{r},n} + x^{\#}_{\mathbf{i},n}\mathbf{i} + x^{\#}_{\mathbf{j},n}\mathbf{j} + x^{\#}_{\mathbf{k},n}\mathbf{k}\right) \in \mathbb{H}^n, \tag{20}$$

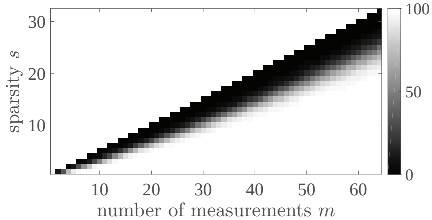
which is the solution of our original problem (6).

The experiments were carried out in MATLAB R2016a on a standard PC machine, with Intel(R) Core(TM) i7-4790 CPU (3.60GHz), 16GB RAM and with Microsoft Windows 10 Pro. The algorithm consisted of the following steps:

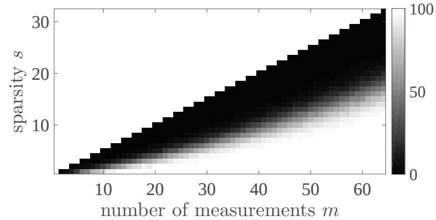
1. Fix constants $n = 256$ (length of $\mathbf{x}$) and $m$ (number of measurements, i.e. length of $\mathbf{y}$) and generate the measurement matrix $\Phi \in \mathbb{H}^{m\times n}$ with Gaussian entries sampled from i.i.d. quaternion normal distribution $\mathcal{N}_{\mathbb{H}}\left(0, \frac{1}{m}\right)$;
2. Choose the sparsity $s \leq \frac{m}{2}$ and draw the support set $S \subseteq \{1,\ldots,n\}$ with $\#S = s$, uniformly at random. Generate a vector $\mathbf{x} \in \mathbb{H}^n$ such that $\operatorname{supp}\mathbf{x} = S$ with i.i.d. quaternion normal distribution $\mathcal{N}_{\mathbb{H}}(0,1)$;
3. Compute $\mathbf{y} = \Phi\mathbf{x} \in \mathbb{H}^m$;
4. Construct vectors $\tilde{\mathbf{y}}, \mathbf{c}$ and matrix $\tilde{\Phi}$ as in (15)–(18);
5. Call the SeDuMi toolbox to solve the problem (14) and calculate the solution $\tilde{\mathbf{x}}^{\#}$;
6. Compute the solution $\mathbf{x}^{\#}$ using (20) and the errors of reconstruction (in the $\ell_1$- and $\ell_2$-norm sense), i.e. $\left\|\mathbf{x}^{\#} - \mathbf{x}\right\|_1$ and $\left\|\mathbf{x}^{\#} - \mathbf{x}\right\|_2$.

The experiment was carried out for $m = 2,\ldots,64$ and $s = 1,\ldots,\frac{m}{2}$. The range of $s$ is not accidental – it is known that, in general, the minimal number $m$ of measurements needed for the reconstruction of an $s$-sparse vector is $2s$ [13, Theorem 2.13]. For each pair of $(m,s)$, we performed 1000 experiments, saving the errors of each reconstruction and the number of perfect reconstructions (the reconstruction is said to be perfect if $\left\|\mathbf{x}^{\#} - \mathbf{x}\right\|_2 \leq 10^{-7}$). For comparison, we also repeated this experiment for the case of $\Phi \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. The percentage of perfect reconstructions in each case is presented in Fig. 1 and Fig. 2 (a).

Fig. 1 (a) presents the dependence of the perfect recovery percentage on the number of measurements $m$ and sparsity $s$ in the quaternion case. We see that simulations confirm our theoretical considerations. Fig. 1 (b) shows the same results for the real case, i.e. $\Phi \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$. Note that in the first experiment for $m = 32$ and $s \leq 9$ the recovery rate is greater than 95%, the same holds for $m = 64$ and $s \leq 20$. It is also worth noticing that the results for corresponding pairs $(m,s)$ are much better in the quaternion setup than in the real-valued case (see Fig. 2(a)). We explain this phenomenon in Lemma 7, namely, we show that for

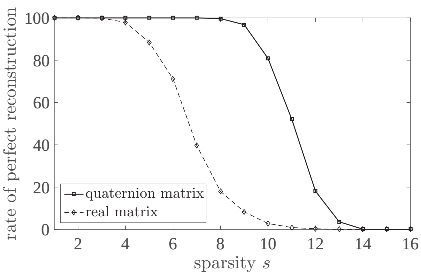



(a) $\Phi \in \mathbb{H}^{m \times n}$

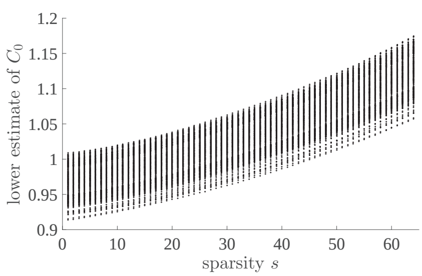



(b) $\Phi \in \mathbb{R}^{m \times n}$

Fig. 1. Results of the recovery experiment for $n = 256$ and different $m$ and $s$. Image intensity stands for the percentage of perfect reconstructions




(a)




(b)

Fig. 2. (a) Comparison of the recovery experiment results for $n = 256$, $m = 32$ and different values of $s$. (b) Lower estimate of the constant $C_0$ in Corollary 6 obtained from the inequality (7) for $n = 256$ and $m = 32$

a fixed vector $\mathbf{x} \in \mathbb{H}^m$ and the ensemble of quaternion Gaussian random matrices $\Phi \in \mathbb{H}^{m \times n}$, the ratio random variable $\frac{\|\Phi \mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2}$ has distribution $\Gamma(2m, 2m)$, i.e., its variance equals $\frac{1}{2m}$, which is four times smaller than in the case of a real vector and real Gaussian matrices of the same size. In other words, a quaternion Gaussian random matrix statistically has smaller restricted isometry constant than its real counterpart.

We also performed another experiment illustrating the approximated reconstruction of non-sparse quaternion vectors from the exact data – as stated in Corollary 6. We fixed constants $n = 256$ and $m = 32$ and generated the measurement matrix $\Phi \in \mathbb{H}^{m \times n}$ with random entries sampled from i.i.d. quaternion normal distribution and 1000 arbitrary vectors $\mathbf{x} \in \mathbb{H}^n$ with standard Gaussian random quaternion entries ($\sigma^2 = 1$), without assuming their sparsity. The above-described algorithm (steps **3.–6.**) was applied to approximately reconstruct the vectors. We used the reconstruction errors $\left\|\mathbf{x}^\# - \mathbf{x}\right\|_1$ to obtain a lower bound on the constant $C_0$ as a function of $s$, for $s = 1, \ldots, 64$, using inequality (7), i.e.,

$$C_0 \geq \frac{\left\|\mathbf{x}^\# - \mathbf{x}\right\|_1}{\left\|\mathbf{x} - \mathbf{x}_s\right\|_1},$$

where $\mathbf{x}_s$ denotes the best $s$-sparse approximation of $\mathbf{x}$. Results of this experiment are shown in Fig. 2 (b) in the form of a scatter plot – each point represents a lower estimate of $C_0$ for one vector $\mathbf{x}$ and sparsity $s$. We see that the dependence on $s$ is monotone, as expected.

# 9. CONCLUSIONS

The results of this article form a theoretical background of the classical compressed sensing methods in the quaternion algebra. We extended the fundamental result of this theory to the full quaternion case, namely we proved that if a quaternion measurement matrix satisfies the RIP with a sufficiently small constant, then it is possible to reconstruct sparse quaternion signals from a small number of their measurements via $\ell_1$-norm minimization. We also estimated the error of the approximated reconstruction of a non-sparse quaternion signal from exact and noisy data. This improves our previous result for real measurement matrices and sparse quaternion vectors [1] and explains success of various numerical experiments in the quaternion setup [2, 17, 30].

The article also answers in affirmative the question about the existence of quaternion matrices satisfying the RIP. We confirm that restricted isometry constants of Gaussian quaternion matrices are small with big probability (and typically smaller than their real counterparts). Together with the aforementioned result (quaternion measurement matrices with small RIP constants allow the exact reconstruction of sparse quaternion vectors), it explains why compressed sensing based experiments in the quaternion algebra work, and brings hope for their wider applications. However, this result in the current form is not sharp. One of the reasons for that is the fact that we used techniques previously applied to the case of real subgaussian random matrices.

There are several possibilities of further research in this field – both in theoretical and applied directions. Among others:

- further refinements of the main results in the quaternion algebra or their extensions to different algebraic structures,
- search for quaternion matrices satisfying the RIP other than Gaussian,
- adjusting reconstruction algorithms to quaternions,
- applications of the theory in practice.

Judging by the number of articles concerning quaternion signal processing published in the last decade we expect that this new branch of compressed sensing will attract attention of even more researchers and will develop considerably.

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# SOME REMARKS ON A SUBCLASS OF LIÉNARD'S EQUATIONS

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**Abstract:** In order to introduce a method of estimating the number of limit cycles in Liénard's systems we analyse an interesting example of a family of Liénard's equations. One can numerically observe an infinite series of saddle-node bifurcations for this family which creates arbitrarily many limit cycles. We study these equations analytically. In order to investigate the existence of periodic solutions we consider the family of linearized equations. One can find the explicit form of the Poincaré map for every equation of this linearized family. We show that for sufficiently small values of parameters there is a similarity between the Poincaré maps for linearized and non-linearized families. It leads us to the conclusion that for any natural number $n$ there exists an equation from the non-linearized family with at least $n$ periodic solutions.
**Keywords:** Liénard equation, limit cycles, perturbation
**Mathematics Subject Classification (2020):** 34E10, 37C27, 37G15, 37M20

## 1. INTRODUCTION

One of the most difficult problems concerning Liénard's equations (for definition see section 2) is to determine the number of limit cycles. Earlier studies were mostly focused on estimating the number and the location of limit cycles for polynomial Liénard's systems only (see [1, 2, 3, 5]). In this paper we study the number of limit cycles for a subclass of Liénard equations given by

$$\ddot{x} + kf'(x)\dot{x} + cx = 0, \tag{1}$$

where $k \in \mathbb{R}$, $c > 0$ and the function $f \in C^2(\mathbb{R})$ is odd (we do not assume that $f$ is a polynomial). Our approach to the analysis of (1) is to study its linear approximation which is much easier to understand. Obviously, this approach raises the following question: how similar are the solutions of this linear approximation to the solutions of equation (1)? In this paper we are trying to answer this question. Our main result is the following theorem:

**Theorem 1** (Main Theorem)**.** *Let us consider a differential equation of the form*

$$\ddot{x} + kf'(x)\dot{x} + x = 0, \tag{2}$$

*where $f \in C^2(\mathbb{R})$ is odd and $k > 0$. We denote the Poincaré map of linear approximation of* (2) *by $\xi$. When we take any compact set $A \subset \mathbb{R}$ such that there is at most finitely many zeros of $\xi$ in $A$ and*

$$\forall y \in A \qquad \xi(y) = 0 \implies \big(y \in \operatorname{int}(A) \quad \wedge \quad \xi'(y) \neq 0\big),$$

*then, on the set A, for sufficiently small values of k, the function $\xi$ is similar (in the sense of definition 10) to the Poincaré map of* (2) *(we denote it by $p_k$).*

A Strict formulation of this theorem along with a proof and the definition of "similarity" can be found in section 4. As an example, in the last section of the paper we consider a specific family of equations given by:

$$\ddot{x} + k(a - \cos(x))\dot{x} + x = 0, \qquad a \in \mathbb{R}, \quad k > 0. \tag{3}$$

It occurs that for the family (3) one can (numerically) observe a series of bifurcations which create any number of limit cycles. By $\zeta$ let us denote the derivative of the solution of (3) with respect to the parameter $k$. The equation describing $\zeta$ is

$$\begin{cases} \dot{\zeta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \zeta + \begin{bmatrix} -ay_0 \sin(t) + \sin y_0 \sin(t) \\ 0 \end{bmatrix}, \\ \zeta(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{cases} \tag{4}$$

where $(0, y_0)$ is the starting point of the trajectory. After some computation one can obtain an analytical form of the Poincaré map for equation (4), which can be expressed in an elegant form by the Bessel function

$$\xi(y_0) = \left(\frac{ay_0}{2} - J_1(y_0)\right)\pi.$$

The main result allows us to claim that there exist parameters $a$, $k$ for which equation (3) has arbitrarily many limit cycles. Anyway, since this result holds only for small values of $k > 0$, it is natural to compare it with a numerical one for some realistic values of $k$. This comparison is also shown in the last section (see figure 2 and 3).

We believe that Theorem 1 is quite an interesting result which shows an interesting connection between the number of limit cycles of Liénard's equation (not necessarily polynomial) and the number of roots of the Poincaré map. Moreover, in the case of problem (3) it presents the elegant connection between the Bessel function and the number of existing limit cycles.

**Structure of the article:**

Our main goal is to find periodic trajectories (if they exist) of equation (1) for arbitrary odd function $f \in C^2(\mathbb{R})$. We will attempt that by studying two different testing functions $p_k$ and $\xi$. The function $p_k$ is created by considering the Poincaré map for equation (1), $\xi$ is based on the linear approximation of (1). Both of them represent the behaviour of trajectories which circle the origin.

In section 2 we start our considerations by discussing Liénard's equation and its properties. Section 3 is concentrated on presenting proper testing functions $p_k$ and $\xi$. The main "similarity" result is presented in section 4 along with a proof which is divided into few lemmas. In the last section we use Theorem 12 to analyse family of equations (3).

## 2. LIÉNARD EQUATION

**Definition 2** (Liénard equation)**.** *A differential equation given by:*

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0, \tag{5}$$

*where $f, g \in C^1(\mathbb{R})$ are odd is called a Liénard's equation.*

**Remark 3.** *The differential equation (5) is equivalent to the system of differential equations given by:*

$$\begin{cases} \dot{x} = -f(x) + y, \\ \dot{y} = -g(x). \end{cases}$$

**Example 4.** *The basic example of a Liénard's equation is the well-known Van der Pol equation:*

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0.$$

*Due to Remark 3 Van der Pol equation can be rewritten as:*

$$\begin{cases} \dot{x} = -\mu\left(\frac{x^3}{3} - x\right) + y, \\ \dot{y} = -x. \end{cases} \tag{6}$$

*One can prove that for $\mu > 0$ system* (6) *has only one periodic solution which is stable (see for example [4]).*

**Remark 5.** *The behaviour of the solutions of* (5) *is similar (by reversing time) to the behaviour of the solutions to*

$$\ddot{x} - f'(x)\dot{x} + g(x) = 0.$$

From now on we only consider the sub-class of Liénard equations in the form

$$\ddot{x} + kf'(x)\dot{x} + cx = 0, \tag{7}$$

where the function $f \in C^2(\mathbb{R})$ is odd, $c > 0$ and $k \in \mathbb{R}$.

**Remark 6.** *If $k \neq 0$ then the behaviour of solutions to* (7) *is similar (by scaling time) to the behaviour of solutions to*

$$\ddot{x} + \frac{k}{\sqrt{c}} f'(x)\dot{x} + x = 0.$$

Since we are only interested in the behaviour of solutions to (7), we can use remarks 3, 5 and 6 to reformulate the problem and consider the following class of differential equations:

$$\begin{cases} \dot{x} = -kf(x) + y, \\ \dot{y} = -x, \end{cases} \tag{8}$$

where $k > 0$ and $f \in C^2(\mathbb{R})$ is odd. In this formulation we omit the case $k = 0$ in (7); however, in this situation the problem becomes trivial and uninteresting.

**Properties of the ordinary differential equations system** (8)**:**

i. The system has only one stationary point $(0,0)$.
ii. (8) is symmetric with respect to the origin. This property allows us to consider a behaviour of trajectories only in the I and IV quadrants.

## 3. TESTING FUNCTIONS FOR SOLUTIONS

We consider system (8) with an arbitrary odd function $f \in C^2(\mathbb{R})$. Let us denote the solution of system (8) for arbitrary given parameter $k = k_0$ and initial condition $z(0, x_0, y_0, k_0) = (x_0, y_0)$ by $z(t, x_0, y_0, k_0)$. The function $z$ can be expressed as follows

$$z(t, x_0, y_0, k_0) =: (z_1(t, x_0, y_0, k_0), z_2(t, x_0, y_0, k_0)).$$

**Definition 7** (Function $p_k$)**.** *For the system (8) with an arbitrary value of k let us define the function*

$$p_k : \mathbb{R}_+ \cup \{0\} \to \mathbb{R} \cup \{\infty\} \quad \textit{such that} \quad p_k(y) = y + \widetilde{y},$$

*where $\widetilde{y} := z_2(t_b(y), 0, y, k)$ and $t_b(y) > 0$ is the first positive time when the trajectory $z(t, 0, y, k)$ of the system (8) intersects the y-axis. If such time does not exist then we take $p_k(y) = \infty$.*

**Interpretation of the values of $p_k(\cdot)$:**

- $p_k(y) < 0$ means that the trajectory starting from the point $(0,y)$ is circling away from the origin.
- $p_k(y) > 0$ means that the trajectory starting from the point $(0,y)$ is circling towards the origin.
- $p_k(y) = 0$ means that the trajectory starting from the point $(0,y)$ is periodic.

**Properties of the function $p_k$:**

i. If $f \in C^1(\mathbb{R})$ then, because of non-intersection of trajectories, we obtain the following:

$$\exists y_0 > 0, \quad p_k(y_0) = \infty \implies \forall y > y_0, \quad p_k(y) = \infty.$$

ii. For one "circulation" of the trajectory starting from the point $(0,y)$ (for $y \geqslant 0$) the point $\widetilde{y}$ is the minimum of the coordinate y on that piece of trajectory.

iii. By applying *The Implicit Function Theorem* one can easily show that $p_k$ is the same class $C^k$ as the function $f$ (on the set where $p_k$ is finite). In our case we have $p_k \in C^2$.

iv. It can be shown that:

(a) There exists $M > 0$ such that the following implication holds:

$$\forall x \geqslant 0, \quad f(x) > -M \implies \forall y \geqslant 0,\ k \geqslant 0 \qquad p_k(y) \geqslant -2Mk.$$

(b) There exists $M > 0$ such that the following implication holds:

$$\forall x \geqslant 0, \quad f(x) < M \implies \forall y \geqslant 0,\ k \geqslant 0 \qquad p_k(y) \leqslant 2Mk.$$

Therefore, if $f$ is bounded then $p_k$ is also bounded.

*Proof.* We present the proof of (a). The analysis of (b) is similar.

Let us consider the testing function given by:

$$\lambda(x,y) = \frac{1}{2}(y+kM)^2 + \frac{x^2}{2}.$$

We are only interested in the part of the trajectories which are in the I and IV quadrants. Hence

$$\begin{aligned}\frac{d\lambda}{dt}(x(t),y(t)) &= (y+kM)\dot{y} + x\dot{x}\\ &= -x(y+kM) + x(-kf(x)+y)\\ &= -kx(M+f(x)) \leqslant 0.\end{aligned}$$

Therefore, $\lambda$ is non-increasing along that part of the trajectories.

Now let us take $y > 0$, the trajectory starting from the point $(0, y)$ and the first positive time $t$ when this trajectory intersect the y-axis. We know that $\lambda(x(0), y(0)) \geqslant \lambda(x(t), y(t))$. This implies

$$(y + kM)^2 \geqslant (\widetilde{y} + kM)^2. \tag{9}$$

From inequality (9) we obtain: $y \geqslant \widetilde{y} \geqslant -y - 2kM$.
Finally, we get $p_k(y) = y + \widetilde{y} \geqslant y - y - 2kM = -2kM$, as required. □

It should be emphasized that the problem of finding the function $p_k$ is at least as difficult as solving the system (8). Because of that $p_k$ cannot be used directly to analyse solutions of our equation and we have to omit the problem somehow.

Let us denote:

$$\zeta(t, x_0, y_0) := \frac{\partial z(t, x_0, y_0, k)}{\partial k}_{|k=0}. \tag{10}$$

Observe that $\zeta$ is a vector valued function. Therefore, we can expressed it as follows:

$$\begin{aligned}(\zeta_1(t, x_0, y_0), \zeta_2(t, x_0, y_0)) &:= \zeta(t, x_0, y_0) \\ &= \left( \frac{\partial z_1(t, x_0, y_0, k)}{\partial k}_{|k=0}, \frac{\partial z_2(t, x_0, y_0, k)}{\partial k}_{|k=0} \right).\end{aligned}$$

**Definition 8** (Function $\xi(y)$)**.** *In respect to earlier denotations let us define $\xi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ by $\xi(y) := \zeta_2(\pi, 0, y)$.*

Due to the property i of system (8) we have that $\xi(0) = 0$.

**Lemma 9.** $\zeta_1(\pi, 0, y) \equiv 0$ *and* $\xi(y) = \zeta_2(\pi, 0, y) = \int_0^\pi \sin(\tau) f(y \sin(\tau)) d\tau$.

*Proof.* By applying *Smoothness of Flows Theorem* to the system (8) we obtain that $z \in C^2$ in respect to all variables. Moreover, the following variational equation is satisfied:

$$\frac{d}{dt} \frac{\partial z}{\partial k}_{|k=k_0}(t, x_0, y_0, k) = D_z \widetilde{f}(z(t, x_0, y_0, k_0), k_0) \frac{\partial z}{\partial k}_{|k=k_0}(t, x_0, y_0, k) + \frac{\partial \widetilde{f}}{\partial k}(z(t, x_0, y_0, k_0), k_0),$$
$$\frac{\partial z}{\partial k}_{|k=k_0}(0, x_0, y_0, k) = 0,$$

where we denote $\widetilde{f}(x, y, k) := \begin{bmatrix} -kf(x) + y \\ -x \end{bmatrix}$ – the right-hand side of the system (8).

Let us take $k_0 = 0$, $x_0 = 0$, $y_0 = y > 0$ and use the notation (10). Hence we obtain the following differential equation:

$$\begin{cases} \dfrac{d\zeta}{dt}(t, 0, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \zeta(t, 0, y) + \begin{bmatrix} -f(y \sin(t)) \\ 0 \end{bmatrix}, \\ \zeta(0, 0, y) = 0. \end{cases} \tag{11}$$

To solve (11) one can use the standard variation of parameters method and obtain

$$\zeta(t,0,y) = \begin{bmatrix} F_1(t)\cos(t) + F_2(t)\sin(t) \\ -F_1(t)\sin(t) + F_2(t)\cos(t) \end{bmatrix},$$

where: $F_1(t) := \int_0^t -\cos(\tau) f(y\sin(\tau))\mathrm{d}\tau$, $F_2(t) := \int_0^t -\sin(\tau) f(y\sin(\tau))\mathrm{d}\tau$.
Now, we take $t = \pi$, which leads to

$$\zeta(\pi,0,y) = \begin{bmatrix} -F_1(\pi) \\ -F_2(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ \int\limits_0^\pi \sin(\tau) f(y\sin(\tau))\mathrm{d}\tau \end{bmatrix},$$

as required. □

Lemma 9 implies that if we consider an infinitesimal value of $k$ then, after time $t = \pi$, the trajectory of system (8) starting from positive part of $y$-axis will not be perturbed in direction $x$. One can think of $\xi$ as the Poincaré map for the linearized system (8).

**Interpretation of the values of function $\xi(\cdot)$:**

- $\xi(y) < 0$ implies that the function $z_2(\pi,0,y,\cdot)$ is decreasing on a sufficiently small interval, which contains $k = 0$ (linearized system). We also know that $z_2(\pi,0,y,0) = -y$ and as a consequence $z_2(\pi,0,y,k) < -y$ for sufficiently small positive values of $k$. In other words, for such values of $k$, the trajectory starting from the point $(0,y)$ should circle away from the origin.

- $\xi(y) > 0$ implies that $z_2(\pi,0,y,\cdot)$ is increasing on a sufficiently small interval, which contains $k = 0$. As a consequence we have that $z_2(\pi,0,y,k) > -y$ for sufficiently small positive values of $k$. In other words, for such values of $k$, the trajectory starting from the point $(0,y)$ should circle towards the origin.

- $\xi(y) = 0$ means, that if the parameter $k$ is infinitesimally small, then after the time $t = \pi$, there should be no disturbance of trajectory of (8) in any direction in relation to the linearized system ($k = 0$), where all trajectories are periodic.

We can observe that in the case of infinitesimally small values of $k$ the interpretation of the values of the function $\xi$ is much the same as for the function $p_k$. What is more, in order to find the explicit expression of $\xi$ we only need to know the form of the function $f$. Our main goal is to show that, indeed there exists a positive value $k$ such that both functions $p_k$ and $\xi$ give us the same behaviour of the trajectories.

# 4. SIMILARITY OF FUNCTIONS $P_K$ AND $\xi$

Next step is to check if functions $p_k$ and $\xi$ are similar in some way. First of all, we have to define how we understand this similarity.

**Definition 10** (Similar functions)**.** *Suppose that $f_1, f_2 \in C([a,b])$ and $[a,b] \subset \mathbb{R}$ is a set where both functions $f_1$ and $f_2$ have exactly $n \in \mathbb{N} \cup \{0\}$ zeros. Denote zeros of function $f_1$ by $y_1, \ldots, y_n$. We say that $f_1$ is similar to $f_2$ on the compact set $[a,b]$ if there exists a homeomorphism $h : [a,b] \to [a,b]$ such that:*

1. *$h(a) = a,\ \ h(b) = b$.*

2. *$f_1(y) \cdot (f_2 \circ h)(y) \geqslant 0$ in $[a,b]$ and the equality holds only for $y = y_i$ (for $i = 1, \ldots, n$).*

*We denote this similarity by $f_1 \underset{[a,b]}{\approx} f_2$.*

**Remark 11.** *Relation $f_1 \underset{[a,b]}{\approx} f_2$ is an equivalence relation.*

**Theorem 12.** *Suppose that $f \in C^2(\mathbb{R})$ is odd and let us consider*

$$\begin{cases} \dot{x} = -kf(x) + y, \\ \dot{y} = -x. \end{cases} \tag{12}$$

*Suppose that for some closed set $[a,b] \subset \mathbb{R}_+$ the function $\xi$ has only finitely many zeros $y_1, \ldots, y_n$ in that set. What is more, $y_i \in (a,b)$ and $\xi'(y_i) \neq 0$ for $i = 1, \ldots, n$. Then, for a sufficiently small positive values of k, we have $\xi \underset{[a,b]}{\approx} p_k$.*

Proof of Theorem 12 is a consequence of the following lemmas:

**Lemma 13.** *Assume that for some compact set $[a,b] \subset \mathbb{R}_+$ (it can be degenerated) there exists a positive value $k_0$ such that one of the following holds:*

1. *For every $k \in [-k_0, k_0]$ and every $y \in [a,b]$ the value of $\frac{\partial z_2}{\partial k}(\pi, 0, y, k)$ is negative.*

2. *For every $k \in [-k_0, k_0]$ and every $y \in [a,b]$ the 1value of $\frac{\partial z_2}{\partial k}(\pi, 0, y, k)$ is positive.*

*Then:*

$$p_k(y) \cdot \frac{\partial z_2}{\partial k}(\pi, 0, y, k) > 0, \qquad \forall k \in (0, k_0], \quad \forall y \in [a,b].$$

*Proof.* We consider two different cases.

**1.** $\quad \forall y \in [a,b], \quad \forall k \in [-k_0,k_0], \quad \frac{\partial z_2}{\partial k}(\pi,0,y,k) < 0.$
Let us take $y_0 \in [a,b]$. Due to our assumption $z_2(\pi,0,y_0,k)$ is decreasing for $k \in [-k_0,k_0]$. In particular, for a fixed $k \in (0,k_0]$

$$-y_0 = z_2(\pi,0,y_0,0) > z_2(\pi,0,y_0,k) \geqslant \widetilde{y_0},$$

where the last inequality holds by property ii of the function $p_k$. Hence

$$p_k(y_0) = y_0 + \widetilde{y_0} < y_0 + (-y_0) = 0.$$

Because we have chosen arbitrary $k$, it leads us to the conclusion that $p_k(y_0) < 0$ for every $k \in (0,k_0]$. Since $y_0$ was chosen arbitrarily this ends the proof of case 1.

**2.** $\quad \forall y \in [a,b], \quad \forall k \in [-k_0,k_0], \quad \frac{\partial z_2}{\partial k}(\pi,0,y,k) > 0.$
Again, we take any $y_0 \in [a,b]$. By the assumption we obtain that $z_2(\pi,0,y_0,k)$ is increasing for $k \in [-k_0,k_0]$. Let us take any $k \in (0,k_0]$. The argument from case 1 cannot be used directly here because the trajectory does not have to intersect y-axis in the time $t = \pi$. It means that the inequality $z_2(\pi,0,y_0,k) \leqslant \widetilde{y_0}$ may not hold.

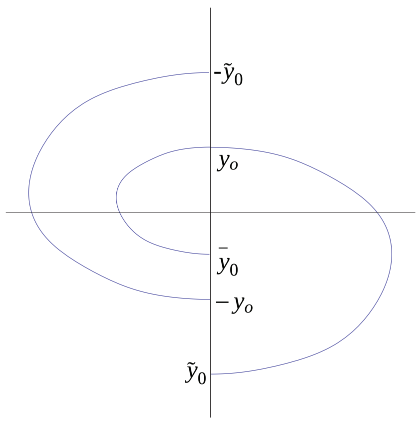


Fig. 1. Example of trajectories of (12) for parameter $-k$

Therefore, let us consider (12) with parameter $-k$. This simple trick yields the following

$$-y_0 = z_2(\pi,0,y_0,0) > z_2(\pi,0,y_0,-k) \geqslant \widetilde{y_0}.$$

By Remark 5, the solutions of system (12) with parameter $k$ obtained by reversing the time are the same as the solutions of (12) with parameter $-k$. Hence, let us denote by $\overline{y_0}$ such value that the trajectory of the system with parameter $-k$ starting from the point $(0,\overline{y_0})$ intersects y-axis for the first time at the point $(0,y_0)$. Observe that, due to earlier remarks, if we consider the system (12) with parameter $k$, then the above notation can be interpreted as $\widetilde{y_0} = \overline{y_0}$. Therefore, our goal is to show that $-y_0 < \overline{y_0}$.

Because of the symmetry of system (12) and the fact that the trajectories do not intersect we obtain the required inequality (see figure 1). Hence, for the system (12) with parameter $k$ it holds that $-y_0 < \overline{y_0} = \widetilde{y_0}$. As a consequence

$$p_k(y_0) = y_0 + \widetilde{y_0} > y_0 + (-y_0) = 0.$$

Arguments similar to those used in case 1 end the proof. □

**Lemma 14.** *Suppose that $y_0 > 0$ is a zero of $\xi$ and $\xi'(y_0) \neq 0$. Then for any neighbourhood $\left[y_0^-, y_0^+\right]$, $0 < y_0^- < y_0 < y_0^+ < \infty$, there exists a positive constant $k_0$ such that $\forall k \in [0, k_0]$ the function $p_k$ has a zero in the neighbourhood $\left(y_0^-, y_0^+\right)$.*

*Proof.* Let us take a neighbourhood $\left[y_0^-, y_0^+\right]$ of the point $y_0$ where $\xi$ has no zeroes other than $y_0$. It can be done because due to our assumptions $\xi$ changes its sign in $y_0$.

Without loss of generality we can assume that $\xi(y_0^-) < 0$ and $\xi(y_0^+) > 0$. In other words, it means that $\frac{\partial z_2}{\partial k}(\pi, 0, y_0^-, 0) < 0$ and $\frac{\partial z_2}{\partial k}(\pi, 0, y_0^+, 0) > 0$. *Smoothness of Flows Theorem* yields $\frac{\partial z_2}{\partial k} \in C^1$ in respect to all variables. Because of that regularity, there exists a positive constant $k_0$ such that for all $k \in [-k_0, k_0]$ we have $\frac{\partial z_2}{\partial k}(\pi, 0, y_0^-, k) < 0$ and $\frac{\partial z_2}{\partial k}(\pi, 0, y_0^+, k) > 0$. Hence, by Lemma 13 we obtain

$$\forall k \in (0, k_0] \qquad p_k(y_0^-) < 0 \quad \wedge \quad p_k(y_0^+) > 0.$$

Finally, continuity of $p_k$ implies that for every $k \in (0, k_0]$ it has a zero in the interval $\left(y_0^-, y_0^+\right)$. □

Let us recall a basic property of continuous functions.

**Lemma 15.** *Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in respect to both variables and $f(\cdot, 0)$ is positive or negative on a closed interval $[a, b] \subset \mathbb{R}$. Then there exists a positive constant $k_0$ such that for every $k \in [-k_0, k_0]$ we have $f(x, k) \cdot f(x, 0) > 0$ for all $x \in [a, b]$.*

**Lemma 16.** *Suppose that $y_0 > 0$ is a zero of $\xi$ and $\xi'(y_0) \neq 0$. Then there exists $a > 0$ such that for every neighbourhood of the form $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$ where $\widetilde{a} \in (0, a]$ there exists $\widetilde{k_0}(\widetilde{a}) > 0$ such that for every $k \in \left(0, \widetilde{k_0}(\widetilde{a})\right]$ the function $p_k$ has exactly one zero in the set $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$.*

*Proof.* Let us choose $a > 0$, small enough such that in the neighbourhood $[y_0 - a, y_0 + a]$ we have $\xi'(y) \neq 0$. It can be done because $\xi'(y_0) \neq 0$ and $\xi \in C^1$. Taking any $\widetilde{a} \in (0, a]$ and using Lemma 14 we get that there exists $k_0(\widetilde{a}) > 0$ such that for every $k \in (0, k_0(\widetilde{a})]$ the function $p_k$ has a zero in the set $(y_0 - \widetilde{a}, y_0 + \widetilde{a})$.

Now, let us take any $x \in [y_0 - \widetilde{a}, y_0 + \widetilde{a}]$ and $\widetilde{k_0}(\widetilde{a}) \in (0, k_0(\widetilde{a})]$ which we will choose later. We want to study the sign of $\frac{\partial p_{k'}}{\partial y}(x)$ for a fixed $k' \in (0, \widetilde{k_0}(\widetilde{a})]$. Observe that under the previous assumptions we have that $p_k \in C^2$ which allows us to use *Schwarz's Theorem*:

$$\frac{\partial p_{k'}}{\partial y}(x) = \frac{\partial p_0}{\partial y}(x) + \int_0^{k'} \frac{\partial}{\partial k}\left(\frac{\partial p_k}{\partial y}(x)\right) \mathrm{d}k = \int_0^{k'} \frac{\partial}{\partial y}\left(\frac{\partial p_k}{\partial k}(x)\right) \mathrm{d}k. \tag{13}$$

Obviously, $p_0(y) \equiv 0$ because for $k = 0$ all trajectories are closed. Observe that (due to the choice of the constant $a$) the function $\frac{\partial}{\partial y}\left(\frac{\partial p_k}{\partial k}_{|k=0}(\cdot)\right) = \frac{\partial \xi}{\partial y}(\cdot)$ has a constant sign in the set $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$. Moreover, the function $\frac{\partial}{\partial y}\left(\frac{\partial p_k}{\partial k}(\cdot)\right)$ is continuous in respect to both variables. From Lemma 15, there exists a positive constant $\widetilde{k_0}(\widetilde{a})$ such that $\widetilde{k_0}(\widetilde{a}) \leqslant k_0(\widetilde{a})$ and the function $\frac{\partial}{\partial y}\left(\frac{\partial p_k}{\partial k}(\cdot)\right)$ has a constant sign in the set $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$ for any $k \in \left[0, \widetilde{k_0}(\widetilde{a})\right]$. Because of that, the last integral in (13) has a constant sign. Moreover, since $x \in [y_0 - \widetilde{a}, y_0 + \widetilde{a}]$ has been chosen freely, for every $k \in (0, \widetilde{k_0}(\widetilde{a})]$ the function $\frac{\partial p_k}{\partial y}(\cdot)$ has a constant sign in $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$. Which means that $p_k$ has only one zero in $[y_0 - \widetilde{a}, y_0 + \widetilde{a}]$. □

**Remark 17.** *By using arguments similar to those in the proof of Lemma 16 one can show that if $\xi'(0) \neq 0$ then there exists a positive constant a such that for every interval $[0, \widetilde{a}]$ where $\widetilde{a} \in (0, a]$ there exists $\widetilde{k_0}(\widetilde{a}) > 0$ such that for all $k \in (0, \widetilde{k_0}(\widetilde{a})]$ the function $p_k(\cdot)$ has exactly one zero in the set $[0, \widetilde{a}]$, which is obviously $y = 0$.*

Let us finally present the proof of Theorem 12. It is worth mentioning that, along with Remark 17, it also gives us the result about similarity of $\xi$ and $p_k$ on the closed sets $[a,b] \subset \mathbb{R}_+ \cup \{0\}$.

*Proof (of Theorem 12).* Let us take a set $[a,b] \subset \mathbb{R}_+$ which satisfies all of the conditions. For every $y_i$ we use Lemma 16 and choose the size of every neighbourhood $U_i$ (of $y_i$) so that they are pairwise disjoint for $i = 1, 2, \ldots, n$. Now we have $n$ neighbourhoods of zeros and in every $U_i$ there exists exactly one zero of $p_k$ if only values of $k$ are smaller than some positive value $\widetilde{k_i}$. We choose $\widetilde{k} := \min\left\{\widetilde{k_1}, \widetilde{k_2}, \ldots, \widetilde{k_n}\right\}$. Obviously, $\widetilde{k}$ is positive and we see that for $k \in (0, \widetilde{k}]$ there is a unique zero of $p_k$ in every neighbourhood $U_i$. Now we apply Lemma 15 and then Lemma 13 to $\frac{\partial z_2}{\partial k}(\pi, 0, y, k)$ and to every connected component of the set $U := \overline{[a,b] \setminus \bigcup_{i=1}^{n} U_i}$. We obtain upper bounds $k_0, k_1, \ldots, k_n > 0$ for such positive $k$ that $p_k(y) \cdot \xi(y) > 0$ in the corresponding set. We take the value $\bar{k} := \min\{k_0, k_1, \ldots, k_n\} > 0$ such that for every $k \in \left(0, \bar{k}\right]$ we have: $p_k(y) \cdot \xi(y) > 0$ in the set $U$. Finally, taking $K := \min\left\{\widetilde{k}, \bar{k}\right\} > 0$ gives that $\xi \underset{[a,b]}{\approx} p_k$ holds for every $k \in (0, K]$. □

# 5. APPLICATION

Let us consider the family of differential equations given by:

$$\ddot{x} + (A + B\cos(x))\dot{x} + Cx = 0, \qquad A, B \in \mathbb{R}, \quad C > 0. \tag{14}$$

By the remarks in section 2 it is enough to consider the following family of differential equations:

$$\ddot{x} + k(a - \cos(x))\dot{x} + x = 0 \qquad a \in \mathbb{R},\ k > 0. \tag{15}$$

In (15) we skipped the trivial case when $B = 0$ because in that case one can find the explicit solutions of equation (14).

Using pretty much the same methods as in the proof of Liénard's Theorem the following generalization can be easily shown.

**Theorem 18** (Generalization of Liénard's Theorem)**.** *Let us consider an ordinary differential equation given by:*

$$\ddot{x} + f'(x)\dot{x} + g(x) = 0, \tag{16}$$

*where $f, g \in C^1(\mathbb{R})$ satisfy the following conditions:*

1. $g(x) \cdot x > 0$ *for all* $x \neq 0$.
2. $f(0) = 0,\ f'(0) < 0$.
3. $\exists\, a^+ > 0, a^- < 0$ *such that* $\liminf_{x\to\infty} f(x) > a^+$ *and* $\limsup_{x\to-\infty} f(x) < a^-$

*Then equation (16) has a periodic solution. Moreover, under the additional assumptions:*

- *the functions $f$, $g$ are odd,*
- *$f(x)$ increases monotonically for $x > \beta$, where $\beta$ is a single positive zero of $f$.*

*equation (16) has exactly one periodic solution and it is stable.*

Theorem 18 allows us to conclude that (15) has a periodic solution for $a \in (0,1)$. Furthermore, by using the Bendixson's negative criterion one can prove that for $|a| \geq 1$ equation (15) has no periodic solutions. Unfortunately, it does not say anything about periodic solutions for $a \in (-1,0]$. To obtain any information about them we are going to use Theorem 12.

Therefore, let us write equation (15) in the form:

$$\begin{cases} \dot{x} = -k(ax - \sin(x)) + y, & a \in \mathbb{R}, \quad k > 0, \\ \dot{y} = -x. \end{cases} \tag{17}$$

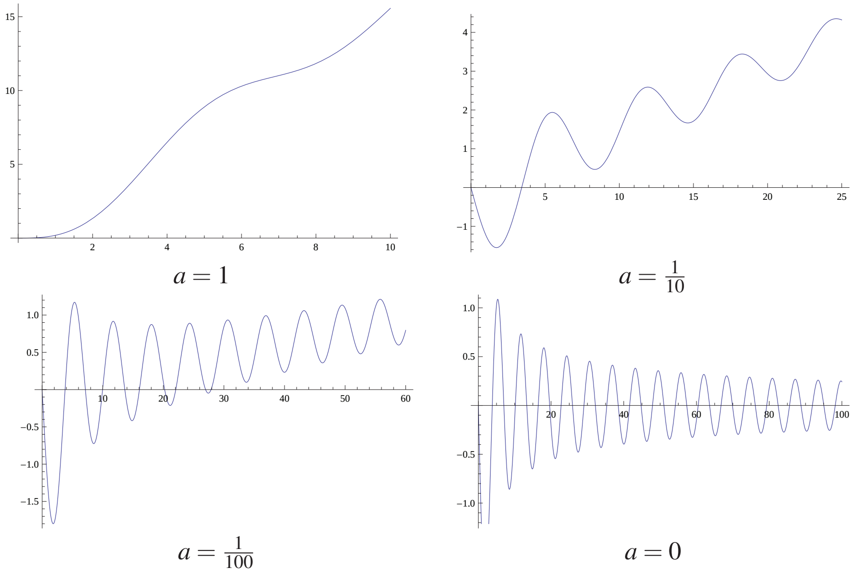


Fig. 2. Function $\xi$ for various values of the parameter $a$

Using Lemma 9 one can obtain that the function $\xi$ has the form:

$$\xi(y) = \left(\frac{ay}{2} - J_1(y)\right)\pi.$$

Below, in figure 2 we present $\xi$ for different values of the parameter $a$.

Obviously, it is possible to choose a parameter $a$ such that for any $n \in \mathbb{N}$ the function $\xi$ has exactly $n$ positive zeros $(y_1, \ldots, y_n)$ and what is more, $\xi'(y_i) \neq 0$ for $i = 1, \ldots, n$. Now, we can choose a positive value $M$ that $\max\{y_1, y_2, \ldots, y_n\} < M$. Theorem 12 and Remark 17 give that for sufficiently small positive values of $k$ the relation $\xi \underset{[a,b]}{\approx} p_k$ is valid in the set $[0, M]$. In other words, for every $n \in \mathbb{N}$ and sufficiently large $M > 0$ one can find parameters $a, k$ such that the function $p_k$ for system (17) has exactly $n$ zeros (periodic solutions) in the set $[0, M]$.

Despite the fact that the above result about similarity of $p_k$ and $\xi$ is obtained only for sufficiently small $k$ it is worth mentioning that $\xi$ seems to remain similar to $p_k$ even for larger values of $k$. As for example in figure 3 we present the function $p_1$ which has been calculated numerically. It suggests that this method of analysis can be helpful as a preliminary assessment of the solutions’ behaviour for these types of Liénard’s equation. Moreover, since this method requires a minimum of computational complexity compared to solving the equation for many different parameters it seems to be cost-effective when $f$ depends on some additional parameters.

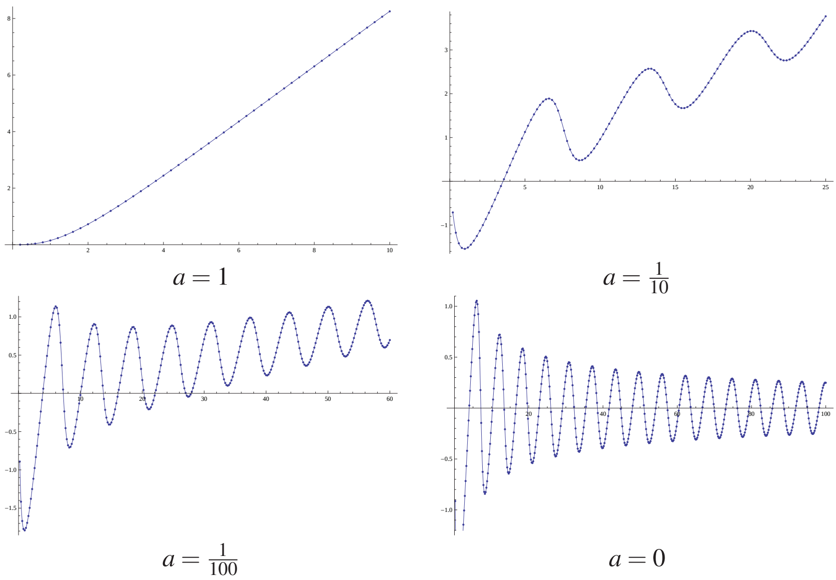


Fig. 3. Function $p_1$ for various values of parameter $a$

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# EXISTENCE OF A WEAK SOLUTION TO KOLMOGOROV'S TWO-EQUATION MODEL OF TURBULENCE IN PERIODIC SETTING

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**Abstract:** We prove the existence of weak solutions to Kolmogorov's two equation model of turbulence in the periodic setting. Kolmogorov's model forms the basis for modern approaches to modelling turbulences such as $k-\varepsilon$, $k-\omega$. The solution is attained by the consideration of several approximate systems and derivations of adequate estimates. The result is inspired by [5], where the authors consider the same system in bounded domain. Additionally, we provide more detailed proof for the reader's convenience.
**Keywords:** weak solution, turbulence, Kolmogorov's two equation model of turbulence, periodic domain
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## 1. INTRODUCTION

Firstly, we provide a short introduction to turbulence modelling and its main idea (see [17], [16], [6]). Next, we explain necessity of establishing additional equations to the model of turbulence. Finally, we introduce Kolmogorov's two equation model and its connection to the currently used turbulence models.

Turbulent flow is a fluid motion characterized by large velocity and pressure gradients, i.e., fluctuations. This causes difficulties in finding solutions using numerical methods. In order to correctly resolve such flow field, dense mesh and very short time steps are required. Thus, the time needed to perform calculations for even relatively simple cases is of the order of weeks, making it inapplicable for commercial simulations.

In the vast majority of industrial simulations, knowledge of the mean flow is sufficient to provide answers to the problems considered. Thus, the simplest idea would be to decrease fluctuations of solutions by considering an average value of velocity and pressure. This is the case in RANS (Reynolds-averaged Navier–Stokes), where an average is taken with respect to time (other plausible averages are, e.g., space average or ensemble average). Now, let us decompose the velocity $v$ and $p$:

$$v(x,t) = \overline{v}(x,t) + \widetilde{v}(x,t), \quad p(x,t) = \overline{p}(x,t) + \widetilde{p}(x,t),$$

where $\overline{v}$, $\overline{p}$ are time-averaged values and $\widetilde{v}$, $\widetilde{p}$ are fluctuations. We substitute the decomposed functions to the Navier Stokes system and we obtain (for details see chapter 2 of [17]):

$$\frac{\partial \overline{v}}{\partial t} + \overline{v} \cdot \nabla \overline{v} - \nu \operatorname{div} D\overline{v} + \nabla \overline{p} = -\operatorname{div}\left(\overline{\widetilde{v} \cdot \widetilde{v}}\right).$$

The last term on the right hand side can be approximated by the Boussinesq approximation (see [17])

$$-\overline{\widetilde{v} \cdot \widetilde{v}} = \nu_T (\nabla \overline{v} + \nabla^T \overline{v}) - \frac{2}{3} kI,$$

where $\nu_T = \frac{k}{\omega}$, $k$ – turbulent kinetic energy, $\omega$ – dissipation rate. Finally, we obtain

$$\frac{\partial \overline{v}}{\partial t} + \overline{v} \cdot \nabla \overline{v} - \nabla \cdot ((\nu + \nu_T) D\overline{v}) + \nabla \left(\overline{p} + \frac{2}{3} k\right) = 0.$$

We see that to close the system, we need to introduce additional equations for $\omega$ and $k$. For details, see [17] and [16].

Nowadays, $k - \varepsilon$ and $k - \omega$ models are most commonly used to calculate $k$ and $\omega$ (for details concerning above-mentioned turbulence models, see [17] and [6]). They bear strong resemblance to Kolmogorov's turbulence model in the way they deal with dissipation, sink and source terms.

In 1941, A.N. Kolmogorov introduced the following system of equations describing turbulent flow ([8], English translation in Appendix A [15]):

$$v_{,t} + \operatorname{div}(v \otimes v) - \nu_0 \operatorname{div}\left(\frac{b}{\omega} D(v)\right) = -\nabla p, \tag{1}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega} \nabla \omega\right) = -\kappa_2 \omega^2, \tag{2}$$

$$b_{,t} + \operatorname{div}(bv) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega} \nabla b\right) = -b\omega + \kappa_4 \frac{b}{\omega} |D(v)|^2, \tag{3}$$

$$\operatorname{div} v = 0, \tag{4}$$

where $v$ – mean velocity, $\omega$ – dissipation rate, $b$ – 2/3 of mean kinetic energy, $p$ – sum of mean pressure and $b$. The novelty of the Kolmogorov formulation lies in the fact that prior knowledge of the length scale (size of large eddies) is no longer required – it can be calculated as $\frac{\sqrt{b}}{\omega}$. The physical motivation of proposed system can be found in [15] and [5]. The mathematical analysis of difficulties in proving the existence of a solution of the system can also be found in [5].

Now, we would like to discus the known mathematical results related to the Kolmogorov's two-equation model of turbulence. There are four recent results devoted to this problem: [5], [11], [10], [9]. In the first one, the authors consider the system in a bounded $C^{1,1}$ domain with mixed boundary conditions for $b$ and $\omega$ and a stick-slip boundary condition for velocity $v$. In order to overcome the difficulties related to the last term on the right hand side of (3), the problem is reformulated and the quantity $E := \frac{1}{2}|v|^2 + \frac{2\nu_0}{\kappa_4}b$ is introduced. Then, the equation (3) is replaced by

$$E_{,t} + \operatorname{div}(v(E+p)) - 2\nu_0 \operatorname{div}\left(\frac{\kappa_3 b}{\kappa_4 \omega}\nabla b + \frac{b}{\omega}D(v)v\right) + \frac{2\nu_0}{\kappa_4}b\omega = 0.$$

The existence of a global-in-time weak solution of the reformulated problem is established.

In [11], the authors consider the system (1)-(4) in periodic domain. It proves the existence of a global-in-time weak solution, but due to the presence of the strongly nonlinear term $\frac{b}{\omega}|D(v)|^2$, the weak form of equation (3) has to be corrected by a positive measure $\mu$, which is zero, provided that a weak solution is sufficiently regular. Some estimates for $\omega$ and $b$ are obtained as well.

In [10], the authors consider the system (1)-(4) in periodic setting in order to show the existence of local strong solutions. Local solutions are obtained, provided that the initial data are in $H^2$. Also, $b$ and $\omega$ are required to be cut off from zero.

In [9], the authors show the existence of global strong solutions of the system (1)-(4), provided that the initial data fulfill the smallness condition. The smallness condition effectively restricts the initial data to ones with small $L^1$ norm of $b_0$ and $L^2$ norm of $v_0$. Additionally, all initial data are required to have small "oscillations" expressed in terms of $L^2$ norms of Laplacians.

The present paper was mainly inspired by [5] and aims to establish analogous result in periodic setting. Additionally, we aim to provide more detailed proofs, making presented arguments more friendly to less experienced readers.

## 2. NOTATION AND MAIN RESULT

Assume that $\Omega = \prod_{i=1}^{3}(0,2\pi)$, $T>0$ and $\Omega^T = \Omega \times (0,T)$. We shall consider the following problem:

$$v_{,t} + \operatorname{div}(v \otimes v) - \nu_0 \operatorname{div}\left(\frac{b}{\omega}D(v)\right) = -\nabla p, \tag{5}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \kappa_1 \operatorname{div}\left(\frac{b}{\omega}\nabla\omega\right) = -\kappa_2\omega^2, \tag{6}$$

$$b_{,t} + \operatorname{div}(bv) - \kappa_3 \operatorname{div}\left(\frac{b}{\omega}\nabla b\right) = -b\omega + \kappa_4\frac{b}{\omega}|D(v)|^2, \tag{7}$$

$$\operatorname{div} v = 0 \tag{8}$$

in $\Omega^T$ with periodic boundary condition on $\partial\Omega$ and the initial condition

$$v_{|t=0} = v_0, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0. \tag{9}$$

Here, $\nu_0, \kappa_1, \ldots, \kappa_4$ are positive constants. For simplicity, we assume that all constants except $\kappa_2$ are equal to one. The reason is that $\kappa_2$ plays an important role in the a priori estimates.
Now, we specify the initial data. Let us assume in a standard way, that the initial condition for velocity field fulfills

$$v_0 \in L^2_{\operatorname{div}}(\Omega). \tag{10}$$

For turbulent kinetic energy, we assume that initially it is as follows:

$$b_0 \in L^1(\Omega), \quad \ln b_0 \in L^1(\Omega), \quad b_0 > 0. \tag{11}$$

Finally, the initial values of frequency $\omega$ are as follows:

$$\omega_0 \in L^\infty(\Omega), \quad 0 < \omega_{\min} \leq \omega_0 \leq \omega_{\max} < \infty. \tag{12}$$

Now, we introduce definitions of function spaces. By $W^{1,r}(\Omega)$, where $r \geq 1$, we denote the space of restrictions to $\Omega$ of the functions, which belong to the space

$$\{u \in W^{1,r}_{loc}(\mathbb{R}^3) : \ u(\cdot + k2\pi e_i) = u(\cdot) \quad \text{for} \quad k \in \mathbb{Z}, \ i = 1,2,3\},$$

where $\{e_i\}_{i=1}^3$ forms the standard basis in $\mathbb{R}^3$. Additionally, we define $W^{1,r}_{\operatorname{div}}(\Omega)$ in the following way:

$$W^{1,r}_{\operatorname{div}}(\Omega) = \{v \in W^{1,r}(\Omega)^3 : \ \operatorname{div} v = 0 \text{ in } \Omega, \int_\Omega v dx = 0\}.$$

Dual spaces of $W^{1,r}$ and $W^{1,r}_{\operatorname{div}}$ will be denoted, respectively, in the following way:

$$W^{-1,r'}(\Omega) := \left(W^{1,r}(\Omega)\right)^*, \qquad W^{-1,r'}_{\operatorname{div}}(\Omega) := \left(W^{1,r}_{\operatorname{div}}(\Omega)\right)^*,$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. By $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, we denote classical norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, \qquad \|f\|_{1,p} = \left(\|f\|_p^p + \sum_{i=1}^3 \|\partial_{x_i} f\|_p^p\right)^{\frac{1}{p}}.$$

Now, we define the following transformation:

$$\langle \cdot, \cdot \rangle : W^{-1,r} \times W^{1,r'} \to \mathbb{R}$$

such that for $f \in W^{-1,r}(\Omega)$ and $g \in W^{1,r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we have

$$\langle f, g \rangle := f(g).$$

Thus, we can define norm in dual spaces of Sobolev spaces:

$$\|f\|_{-1,r} = \sup_{\varphi \in W^{1,r'}(\Omega): \|\varphi\|_{W^{1,r'}(\Omega)} = 1} |\langle f, \varphi \rangle| .$$

Also, for $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we define $(\cdot, \cdot)$ in the following way:

$$(f, g) = \int_\Omega f(x) g(x) dx.$$

Additionally, we define

$$L^r(\Omega) := \overline{W^{1,r}(\Omega)}^{\|\cdot\|_r}, \qquad L^2_{\text{div}}(\Omega) := \overline{W^{1,2}_{\text{div}}(\Omega)}^{\|\cdot\|_2}$$

and

$$L^r_0(\Omega) := \{v \in L^r(\Omega) : \int_\Omega v dx = 0\}.$$

Finally, we define the space that will be useful for considerations related to kinetic turbulent energy $b$

$$\begin{aligned} \varepsilon = \{ & b \in L^\infty\left(0, T, L^1(\Omega)\right) : b > 0 \text{ almost everywhere in } \Omega^T, \\ & \ln b \in L^\infty(0, T, L^1(\Omega)), \\ & b \in L^\lambda(0, T, W^{1,\lambda}(\Omega) \forall \lambda \in [1, 2)\}. \end{aligned}$$

Now, we are ready to present the main theorem, which states the existence result to system (5)-(8).

**Theorem 1.** *Let us assume that the initial data satisfy (10)-(12). Then, there exists a quadruple* $(v, b, \omega, p)$ *such that*

$$v \in L^2(0, T, W^{1,2}_{\text{div}}(\Omega)) \cap W^{1,q}(0, T, W^{-1,q}(\Omega)) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{13}$$

$$b \in \varepsilon, \tag{14}$$

$$\partial_t b \in \mathcal{M}\left(0, T, W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{8}{7}\right), \tag{15}$$

$$p \in L^q\left(0, T, L^q_0(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{16}$$

$$E \in W^{1,q}\left(0, T, W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{80}{79}\right), \tag{17}$$

$$b\omega \in L^q(0, T, W^{1,q}(\Omega)) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{18}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leq \omega \leq \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \quad \textit{almost everywhere in } \Omega^T, \tag{19}$$

*where*

$$E := \frac{|v|^2}{2} + b. \tag{20}$$

*In addition, pressure p can be decomposed as* $p = p_1 + p_2$, *where*

$$p_1 \in L^q\left(0,T,L^q_0(\Omega)\right) \textit{ for all } q \in \left[1, \frac{16}{11}\right), \tag{21}$$

$$p_2 \in L^{5/3}\left(0,T,L^{5/3}_0(\Omega)\right). \tag{22}$$

*After denoting*

$$\mu := \frac{b}{\omega}, \tag{23}$$

*the quadruple* $(v,b,\omega,p)$ *satisfies the following identities:*

$$\int_0^T \langle v_{,t},w\rangle - (v\otimes v,\nabla w) + (\mu D(v),D(w))\,dt = \int_0^T (p,\operatorname{div} w)\,dt \qquad \forall w \in L^\infty\left(0,T,W^{1,\infty}(\Omega)\right), \tag{24}$$

$$\int_0^T \langle \partial_t E,z\rangle - (v(E+p),\nabla z) + (\mu\nabla b,\nabla z) + (\mu D(v)v,\nabla z)\,dt = -\int_0^T (b\omega,z)\,dt \qquad \forall z \in L^\infty\left(0,T,W^{1,\infty}(\Omega)\right), \tag{25}$$

$$\int_0^T \langle \partial_t\omega,z\rangle - (v\omega,\nabla z) + \left(\frac{\nabla(b\omega)}{\omega} - \nabla b,\nabla z\right)dt = -\kappa_2\int_0^T \left(\omega^2,z\right)dt \qquad \forall z \in L^\infty\left(0,T,W^{1,\infty}(\Omega)\right), \tag{26}$$

*with the initial data fulfilling*

$$\lim_{t\to 0^+} \|v(t)-v_0\|_2 + \|\omega(t)-\omega_0\|_2 + \|b(t)-b_0\|_1 = 0. \tag{27}$$

*Moreover, the following inequality holds:*

$$\int_0^T \langle b_{,t},z\rangle + (\mu\nabla b,\nabla z) - (vb,\nabla z)\,z dt \geq \int_0^T \left(-b\omega + \mu|D(v)|^2,z\right)dt \qquad \forall z \in C\left(0,T,W^{1,\infty}(\Omega)\right) \textit{ such that } z\geq 0 \textit{ almost everywhere in } \Omega^T. \tag{28}$$

In order to prove above result, we will establish several existence results to auxiliary problems, which approximate the problem (5)-(8). Using established estimates in those approximations, it will be plausible to obtain the existence result of the considered system. Now, we will focus on outlining auxiliary lemmas and notation.

# 3. AUXILIARY RESULTS AND ADDITIONAL NOTATION

Firstly, let us recall that $\Omega = \prod_{i=1}^{3}(0,2\pi)$, $T > 0$ and $\Omega^T = \Omega \times (0,T)$. To define approximate problems, we need to define the cut-off function

$$T_m(s) = \begin{cases} s & \text{if } |s| \le m \\ m\,\mathrm{sgn}(s) & \text{if } |s| > m \end{cases}. \tag{29}$$

Now, we define the function $\Theta_m$, which is the primitive function of $T_m$:

$$\Theta_m(s) := \int_0^s T_m(\tau)d\tau. \tag{30}$$

Next, we consider a smooth, non-increasing function $G$, such that $G(s) = 1$ when $s \in [0,1]$ and $G(s) = 0$ for $s \ge 2$. For $m \in \mathbb{R}_+$, we define

$$G_m(s) := G\left(\frac{s}{m}\right) \tag{31}$$

and we denote

$$\Gamma_m(s) := \int_0^s G_m(\tau)d\tau. \tag{32}$$

In order to avoid confusion, we define $z_+ = \max\{z,0\}$ and $z_- = \min\{z,0\}$.

Additionally, by $\{w_i\}_{i=0}^{\infty}$ we denote an orthogonal basis of $W^{1,2}_{\mathrm{div}}(\Omega)$, which is also orthogonal in $L^2_{\mathrm{div}}$ (such a basis exists due to Lemma 6). By $\{z_i\}_{i=0}^{\infty}$, we denote an orthogonal basis of $W^{1,2}(\Omega)$, which is also an orthogonal in $L^2(\Omega)$.

In the proof of the main theorem, we will need to reconstruct pressure. The following lemma will enable us to do so:

**Lemma 2** (see Lemma C.1 in [3]). *Let $q, q' \in (1,\infty)$, and such that $\frac{1}{q} + \frac{1}{q'} = 1$. Then, there exists linear, bounded operator*

$$\mathcal{L} : L^q(\Omega)^{3\times 3} \to L^q(\Omega), \tag{33}$$

*such that for all $\varphi \in W^{2,q'}(\Omega)$ and any fixed $B \in L^q(\Omega)^{3\times 3}$ the following relation holds:*

$$(\mathcal{L}(B), \Delta\varphi) = \left(B, \nabla^2\varphi\right), \qquad \int_\Omega \mathcal{L}(B)dx = 0. \tag{34}$$

*Proof.* For $B \in \mathcal{D}(\Omega)^{3\times 3}$ we set the system

$$\Delta\mathcal{L}(B) = \mathrm{div}\,\mathrm{div}\,B \quad \text{in } \Omega, \tag{35}$$

$$\int_\Omega \mathcal{L}(B)dx = 0 \tag{36}$$

equipped with periodic boundary conditions. From classical theory of Poisson equation, the solution to system (35)-(36) exists and is smooth. Thus, we can write $\mathcal{L}(B) := (\Delta)^{-1} \operatorname{div}\operatorname{div} B$. Operator $\mathcal{L}$ is linear and continuous as a mapping from $W^{1,q}(\Omega)^{3\times 3}$ to $W^{1,q}(\Omega)$ for all $q \in (1,\infty)$. We also see that multiplying the equation (35) by arbitrary $\varphi \in W^{2,q'}(\Omega)$, and integrating by parts four times, we get (34). Now, we focus on showing boundedness of operator $\mathcal{L} : L^q(\Omega)^{3\times 3} \cap \mathcal{D}(\Omega)^{3\times 3} \to L^q(\Omega)$. To do this, we need to find space-periodic $\varphi$ such that

$$\Delta\varphi = |\mathcal{L}(B)|^{q-2}\mathcal{L}(B) - \frac{1}{|\Omega|}\int_\Omega |\mathcal{L}(B)|^{q-2}\mathcal{L}(B)dx \quad \text{in } \Omega, \tag{37}$$

$$\int_\Omega \varphi dx = 0. \tag{38}$$

From $L^q$ theory for Poisson equation, there exists a constant $C > 0$ depending only on $\Omega$ and $q$ such that

$$\begin{aligned} \int_\Omega |\nabla^2\varphi|^{q'}dx &\le C\int_\Omega \left| |\mathcal{L}(B)|^{q-2}\mathcal{L}(B) - \frac{1}{|\Omega|}\int_\Omega |\mathcal{L}(B)|^{q-2}\mathcal{L}(B)dx\right|^{q'} dx \\ &\le C\int_\Omega |\mathcal{L}(B)|^q dx, \end{aligned} \tag{39}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since $B$ is smooth, the integral on the right-hand side is finite for any $q \in (1,\infty)$. Now, plugging (37) into (34) we get using of the fact that $\int_\Omega \mathcal{L}(B)dx = 0$ and (39), the following inequality:

$$\int_\Omega |\mathcal{L}(B)|^q dx = \left(B, \nabla^2\varphi\right) \le \|B\|_q \left\|\nabla^2\varphi\right\|_{q'} \le C\|B\|_q \|\mathcal{L}(B)\|_q^{q-1}.$$

We obtained $\|\mathcal{L}(B)\|_q \le C\|B\|_q$ for $B \in \mathcal{D}(\Omega)^{3\times 3}$. Since $\mathcal{D}(\Omega)^{3\times 3}$ is a dense subset of $L^q(\Omega)^{3\times 3}$, the operator can be uniquely extended to $\mathcal{L} : L^q(\Omega)^{3\times 3} \to L^q(\Omega)$. Moreover, the system (34) can be established for $B \in L^q(\Omega)^{3\times 3}$ by considering a sequence of smooth $\{B^n\}$ such that $B^n \to B$ in $L^q(\Omega)^{3\times 3}$ and applying weak convergence. This completes the proof. □

For the completeness of presented arguments, we recall Div-Curl lemma.

**Lemma 3** (see:[14, 12] )**.** *Let $\Omega$ be an open set of $R^N$, $N \ge 2$. Let $w$ be a function such that $w : \mathbb{R}^N \to \mathbb{R}$. We denote*

$$\operatorname{div}(w) = \sum_{i=1}^n \frac{\partial w_i}{\partial x_i}, \qquad C_{ij}(w) = \frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}.$$

*Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $n$, let $a^n \in [L^p(\Omega)]^N$, $b^n \in [L^q(\Omega)]^N$ with properties*

$$a^n \overset{n\to\infty}{\rightharpoonup} a \quad \text{weakly in } [L^p(\Omega)]^N,$$

$$b^n \overset{n\to\infty}{\rightharpoonup} b \quad \text{weakly in } [L^q(\Omega)]^N,$$

$$\{\operatorname{div}(a^n)\}_{n=0}^{\infty} \quad \textit{lies in compact subset of } W^{-1,p}(\Omega) \tag{40}$$
$$\{C(b^n)\}_{n=0}^{\infty} \quad \textit{lies in compact subset of } W^{-1,q}(\Omega)^{N\times N}. \tag{41}$$

*Then,*

$$a^n b^n \overset{n\to\infty}{\rightharpoonup} ab \quad \textit{in sense of distributions}.$$

Now, let us formulate a simple corollary of the Vitali convergence lemma.

**Lemma 4** (see Corollary 4.5.5 in [1])**.** *Let $\Omega \subset \mathbb{R}^N$ be bounded and $u_n : \Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$ for some $p > 1$. Suppose that*

1. *$u_n \to u$ almost everywhere in $\Omega$,*
2. *the sequence $u_n$ is bounded in $L^p(\Omega)$.*

*Then,*

$$u_n \to u \quad \textit{in } L^r(\Omega) \textit{ for all } 1 \le r < p.$$

**Lemma 5.** *Let $p, q \in (1, \infty)$ such that $\frac{1}{q} + \frac{1}{p} < 1$. Also, we assume that*

$$u_n \rightharpoonup u \quad \textit{weakly in } L^p(\Omega) \qquad \textit{and} \qquad v_n \to v \quad \textit{strongly in } L^q(\Omega).$$

*Then,*

$$u_n v_n \rightharpoonup uv \quad \textit{weakly in } L^s(\Omega),$$

*where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$.*

*Proof.* Let $s'$ be such that $\frac{1}{s'} + \frac{1}{s} = 1$. We see that $\frac{1}{s'} + \frac{1}{p} + \frac{1}{q} = 1$. Additionally, let $\varphi \in L^{s'}(\Omega)$. Then we have

$$\int_\Omega u_n v_n \varphi dx = \int_\Omega u_n (v_n - v) \varphi dx + \int_\Omega u_n v \varphi dx.$$

The first integral's limit is zero due to strong convergence of $v_n$, boundedness of $u_n$ and the following inequality:

$$\left| \int_\Omega u_n (v_n - v) \varphi dx \right| \le \|u_n\|_p \|v_n - v\|_q \|\varphi\|_{s'} \overset{n\to\infty}{\longrightarrow} 0.$$

Due to the fact that $v\varphi \in L^{p'}(\Omega)$, where $\frac{1}{p'} = \frac{1}{s'} + \frac{1}{q}$, and weak convergence of $u_n$, we have

$$\int_\Omega u_n v \varphi dx \to \int_\Omega uv\varphi dx.$$

This completes the proof of the lemma. $\square$

**Lemma 6** (see: Theorem 2.24 in [13]). *There exists a family of functions* $\mathcal{N} = \{a_1, a_2, a_3, \dots\}$ *such that*

- $\mathcal{N}$ *is an orthonormal basis in* $L^2_{\mathrm{div}}(\Omega)$,
- $a_j \in C^\infty(\overline{\Omega})$,
- $\mathcal{N}$ *is an orthogonal basis in* $W^{1,2}_{\mathrm{div}}(\Omega)$.

**Lemma 7** (Aubin–Lions–Simon, see: Theorem II.5.16 in [2]). *Let* $B_0 \subset B_1 \subset B_2$ *be three Banach spaces. We assume that the embedding of* $B_1$ *in* $B_2$ *is continuous and that the embedding of* $B_0$ *in* $B_1$ *is compact. Let* $p$, $r$ *be such that* $1 \le p, r \le \infty$. *For* $T > 0$, *we define*

$$E_{p,r} = \left\{ v \in L^p(0,T,B_0),\ \frac{dv}{dt} \in L^r(0,T,B_2) \right\}.$$

*i) If* $p < \infty$, *the embedding of* $E_{p,r}$ *in* $L^p(0,T,B_1)$ *is compact.*
*ii) If* $p = \infty$ *an if* $r > 1$, *the embedding of* $E_{p,r}$ *in* $C^0(0,T,B_1)$ *is compact.*

**Lemma 8** (see: chapter 1.2.b in [7]). *Let* $X$ *be Banach space,* $T > 0$ *and* $1 \le p \le \infty$. *Let* $f_n \to f$ *in* $L^p(0,T,X)$. *Then, there exists a subsequence* $f_{n_k}$ *such that* $f_{n_k} \to f$ *in* $X$ *almost everywhere.*

# 4. K-APPROXIMATION

In order to prove Theorem 1, we will establish a series of existence results to approximate problems. We consider following problem:

$$v_{,t} + \mathrm{div}(G_k(|v|^2)\, v \otimes v) - \mathrm{div}\,(T_k(\mu)\, D(v)) = -\nabla p, \tag{42}$$

$$\omega_{,t} + \mathrm{div}(\omega v) - \mathrm{div}\left(\frac{b}{\omega}\nabla\omega\right) = -\kappa_2 \omega^2, \tag{43}$$

$$b_{,t} + \mathrm{div}(b v) - \mathrm{div}\left(\frac{b}{\omega}\nabla b\right) = -b\omega + T_k(\mu)|D(v)|^2, \tag{44}$$

$$\mathrm{div}\, v = 0, \tag{45}$$

in $\Omega^T$, where $\mu = \frac{b}{\omega}$. The system is equipped with periodic boundary condition and the following initial condition:

$$v_{|t=0} = v_0, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^k(x) = b_0(x) + \frac{1}{k}. \tag{46}$$

The following theorem states the existence result for this system:

**Theorem 9.** *Let us fix $k \in \mathbb{N}_+$. Then, there exists a triple $(v, b, \omega)$ such that*

$$v \in L^2(0,T,W^{1,2}_{\mathrm{div}}(\Omega)) \cap W^{1,2}(0,T,W^{-1,2}_{\mathrm{div}}(\Omega)), \tag{47}$$

$$b \in L^q\left(0,T,W^{1,q}(\Omega)\right) \cap L^\infty\left(0,T,L^1(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{5}{4}\right), \tag{48}$$

$$\partial_t b \in L^1\left(0,T,W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{80}{79}\right), \tag{49}$$

$$\omega \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{50}$$

$$\partial_t \omega \in L^q\left(0,T,W^{-1,q}(\Omega)\right) \qquad \text{for all } q \in \left[1, \frac{16}{11}\right), \tag{51}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \le \omega \le \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \quad \text{almost everywhere in } \Omega^T, \tag{52}$$

$$\frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}} \le b, \quad \text{almost everywhere in } \Omega^T, \tag{53}$$

*which solves the problem (42)-(46) in the following sense:*

$$\begin{aligned}\langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w\right) + \left(T_k(\mu) D(v), D(w)\right) = 0& \\ \forall w \in W^{1,2}_{\mathrm{div}}(\Omega) \text{ a.a. } t \in (0,T),&\end{aligned} \tag{54}$$

$$\begin{aligned}\langle b_{,t}, z\rangle - (bv, \nabla z) + (\mu\nabla b, \nabla z) = \left(-b\omega + T_k(\mu)\,|D(v)|^2, z\right)& \\ \forall z \in W^{1,\infty}(\Omega) \text{a.a. } t \in (0,T),&\end{aligned} \tag{55}$$

$$\begin{aligned}\langle \omega_{,t}, z\rangle - (\omega v, \nabla z) + (\mu\nabla \omega, \nabla z) = -\kappa_2\left(\omega^2, z\right)& \\ \forall z \in W^{1,\infty}(\Omega) \text{a.a. } t \in (0,T),&\end{aligned} \tag{56}$$

*where $\mu = \frac{b}{\omega}$. The initial data are attained strongly in the following sense:*

$$\lim_{t\to 0^+} \|v(t) - v_0\|_2 + \|\omega(t) - \omega_0\|_2 + \|b(t) - b_0^k\|_1 = 0. \tag{57}$$

*Moreover, for all $\lambda \in (0,1]$, the following (k-independent) estimate holds:*

$$\begin{aligned}\sup_{t\in(0,T)} &\left(\|b(t)\|_1 + \|\ln b(t)\|_1 + \|v(t)\|_2^2\right) + \int_{\Omega^T} \left(1+b^{-1}\right) T_k(\mu)|D(v)|^2 dxdt \\ &+ \int_{\Omega^T} \frac{\mu}{b^{1+\lambda}}\,|\nabla b|^2 + \mu\,|\nabla\omega|^2 + \mu^{\frac{8}{3}-\lambda} dxdt \\ &\le C(\lambda^{-1}, v_0, b_0, \omega_0, \omega_{\min}, \omega_{\max}).\end{aligned} \tag{58}$$

*Moreover, the following inequality holds for almost all times $t \in (0,T)$:*

$$\begin{aligned}\left(\sqrt{b(t)}, \varphi\right) - \int_0^t \left(\sqrt{b}v, \nabla\varphi\right) d\tau + \int_0^t \left(\frac{\sqrt{b}}{2\omega}\nabla b, \nabla\varphi\right) d\tau& \\ \ge \frac{1}{2}\int_0^t \left(\sqrt{b}\omega, \varphi\right) d\tau + \left(\sqrt{b_0^k}, \varphi\right) \quad \forall \varphi \in D(\Omega), \varphi \ge 0.&\end{aligned} \tag{59}$$

# 5. (N,K)-APPROXIMATION

In order to prove Theorem 9, we introduce the next level of approximation. To define this approximate problem, we first smooth out the initial conditions for $b$ and $v$. First, we find a sequence of smooth, non-negative functions $b_0^n$ such that

$$b_0^n \to b_0 \text{ strongly in } L^1(\Omega) \tag{60}$$

and define

$$b_0^{n,k} := b_0^n + \frac{1}{k}. \tag{61}$$

Now, let $\{w_i\}_{i=0}^{\infty}$ be a smooth basis of $W^{1,2}_{\mathrm{div}}(\Omega)$ that is orthonormal in $L^2(\Omega)$. It exists due to Lemma 6. Using the chosen basis, we introduce the approximated initial condition for velocity $v^n$ as

$$v_0^n = \sum_{i=0}^{n} (v_0, w_i) w_i.$$

Having approximated initial conditions, we consider the following problem:

$$v_{,t} + \mathrm{div}(G_k\left(|v|^2\right) v \otimes v) - \mathrm{div}\left(T_k\left(\mu^n\right) D(v)\right) = -\nabla p, \tag{62}$$

$$\omega_{,t} + \mathrm{div}(\omega v) - \mathrm{div}\left(T_n\left(\mu^n\right) \nabla \omega\right) = -\kappa_2 \omega^2, \tag{63}$$

$$b_{,t} + \mathrm{div}(bv) - \mathrm{div}\left(T_n\left(\mu^n\right) \nabla b\right) = -b\omega + \frac{T_k(\mu^n)|D(v)|^2}{1+n^{-1}|D(v)|^2}, \tag{64}$$

$$\mathrm{div}\, v = 0 \tag{65}$$

in $\Omega^T$, where

$$\mu^n = \frac{b}{\omega} + \frac{1}{n}. \tag{66}$$

Additionally, the problem is equipped with periodic boundary condition and the following initial condition:

$$v_{|t=0} = v_0^n, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^{n,k}(x). \tag{67}$$

The following theorem states the existence result for this system:

**Theorem 10.** *Let us fix $k \in \mathbb{N}_+$, $n \in \mathbb{N}_+$. Then, there exists a triple $(c, b, \omega)$ such that*

$$c \in W^{1,\infty}(0,T)^n, \tag{68}$$

$$b \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^2(\Omega)\right), \tag{69}$$

$$\partial_t b \in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{70}$$

$$\omega \in L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{71}$$

$$\partial_t \omega \in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{72}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t} \leq \omega \leq \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \qquad \textit{almost everywhere in } \Omega^T, \tag{73}$$

$$\frac{1}{k\left(1+\kappa_2\omega_{\min}t\right)^{\frac{1}{\kappa_2}}} \leq b \qquad \textit{almost everywhere in } \Omega^T, \tag{74}$$

*which solves the problem (62)-(67) in the following sense:*

$$\begin{aligned}(v_{,t},w_i) - \left(G_k\left(|v|^2\right)v\otimes v,\nabla w_i\right) + \left(T_k\left(\mu^n\right)D(v),D(w_i)\right) &= 0 \\ &\textit{for all } i=1,\dots,n,\end{aligned} \tag{75}$$

$$\begin{aligned}\langle b_{,t},z\rangle - (bv,\nabla z) + (T_n(\mu^n)\nabla b,\nabla z) + (b\omega,z) &= \left(\frac{T_k\left(\mu^n\right)|D(v)|^2}{1+n^{-1}|D(v)|^2},z\right) \\ &\forall z\in W^{1,2}(\Omega) a.a.\ t\in(0,T),\end{aligned} \tag{76}$$

$$\begin{aligned}\langle \omega_{,t},z\rangle - (\omega v,\nabla z) + (T_n(\mu^n)\nabla\omega,\nabla z) &= -\kappa_2\left(\omega^2,z\right) \\ &\forall z\in W^{1,2}(\Omega) a.a.\ t\in(0,T),\end{aligned} \tag{77}$$

*where* $\mu^n = \frac{b}{\omega}+\frac{1}{n}$ *and*

$$v(t,x) := \sum_{i=1}^{n} c_i(t)w_i(x). \tag{78}$$

*The initial data are attained strongly in the following sense:*

$$\lim_{t\to 0^+} \|v(t)-v_0^n\|_2 + \|\omega(t)-\omega_0\|_2 + \|b(t)-b_0^{n,k}\|_1 = 0. \tag{79}$$

*Moreover, the following inequality holds for almost all times* $t\in(0,T)$*:*

$$\begin{aligned}\left(\sqrt{b(t)},\varphi\right) - \int_0^t\left(\sqrt{b}v,\nabla\varphi\right)d\tau + \int_0^t\left(T_n(\mu^n)\nabla\sqrt{b},\nabla\varphi\right)d\tau \\ \geq -\frac{1}{2}\int_0^t\left(\sqrt{b}\omega,\varphi\right)d\tau + \left(\sqrt{b_0^{n,k}},\varphi\right) \quad \forall\varphi\in D(\Omega),\varphi\geq 0.\end{aligned} \tag{80}$$

# 6. (M,N,K)-APPROXIMATION

Once again, we will approximate the initial data for turbulent kinetic energy. We introduce $b_0^{m,n,k}$ in the following way:

$$b_0^{m,n,k} = \sum_{i=0}^{m}(b_0^{n,k},z_i)z_i,$$

where $\{z_i\}_{i=0}^{\infty}$ denotes the orthogonal basis of $W^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$. Using this, we consider the following problem:

$$v_{,t} + \operatorname{div}(G_k\left(|v|^2\right) v\otimes v) - \operatorname{div}\left(T_k\left(\mu^{n,m}\right) D(v)\right) = -\nabla p, \tag{81}$$

$$\omega_{,t} + \operatorname{div}(\omega v) - \operatorname{div}\left(T_n\left(\mu^{n,m}\right)\nabla\omega\right) = -\kappa_2\omega^2, \tag{82}$$

$$b_{,t} + \operatorname{div}(bv) - \operatorname{div}\left(T_n\left(\mu^{n,m}\right)\nabla b\right) = -b_+\omega + \frac{T_k(\mu^{n,m})|D(v)|^2}{1+n^{-1}|D(v)|^2}, \tag{83}$$

$$\operatorname{div} v = 0 \tag{84}$$

in $\Omega^T$, where $\mu^{n,m} = \frac{b_+}{\omega+\frac{1}{m}} + \frac{1}{n}$. Additionally, the problem is equipped with the periodic boundary condition and the following initial condition:

$$v_{|t=0} = v_0^n, \quad \omega_{|t=0} = \omega_0, \quad b_{|t=0} = b_0^{m,n,k}(x). \tag{85}$$

The following theorem states the existence result for this system:

**Theorem 11.** *Let us fix $k\in\mathbb{N}_+$, $n\in\mathbb{N}_+$ and $m\in\mathbb{N}_+$ such that $m\geq\omega_{\max}$. Then, there exists a triple $(c,d,\omega)$ such that*

$$c\in W^{1,\infty}(0,T)^n, \tag{86}$$

$$d\in W^{1,\infty}(0,T)^m, \tag{87}$$

$$\omega\in L^2\left(0,T,W^{1,2}(\Omega)\right)\cap L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{88}$$

$$\partial_t\omega\in L^2\left(0,T,W^{-1,2}(\Omega)\right), \tag{89}$$

$$\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\leq\omega\leq\frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t} \quad \textit{almost everywhere in } \Omega^T, \tag{90}$$

*which solves the problem (81)-(85) in the following sense:*

$$(v_{,t},w_i) - \left(G_k\left(|v|^2\right)v\otimes v,\nabla w_i\right) + \left(T_k\left(\mu^{n,m}\right)D(v),D(w_i)\right) = 0 \quad \textit{for all } i=1,\dots,n, \tag{91}$$

$$(\partial_t b,z_i) - (bv,\nabla z_i) + (T_n(\mu^{n,m})\nabla b,\nabla z_i) + (b_+\omega,z_i) = \left(\frac{T_k\left(\mu^{n,m}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2},z_i\right) \quad \textit{for all } i=1,\dots,m, \tag{92}$$

$$\langle\omega_{,t},z\rangle - (\omega v,\nabla z) + (T_n(\mu^{n,m})\nabla\omega,\nabla z) + \kappa_2\left(\omega^2,z\right) = 0 \quad \forall z\in W^{1,2}(\Omega) \textit{ a.a. } t\in(0,T), \tag{93}$$

*where*

$$\mu^{n,m} = \frac{b_+}{\omega+\frac{1}{m}} + \frac{1}{n}, \tag{94}$$

$$v(t,x) := \sum_{i=1}^{n} c_i(t)w_i(x), \tag{95}$$

$$b(t,x) := \sum_{i=1}^{m} d_i(t) z_i(x). \tag{96}$$

*The initial data are attained strongly in the following sense:*

$$\lim_{t\to 0^+} \|v(t)-v_0^n\|_2 + \|\omega(t)-\omega_0\|_2 + \|b(t)-b_0^{m,n,k}\|_2 = 0. \tag{97}$$

# 7. PROOF OF THEOREM 9

*Proof of theorem 11.* Let us recall that $\{z_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ denote bases of $W^{1,2}(\Omega)$ and $W^{1,2}_{\mathrm{div}}(\Omega)$, which are orthogonal in $L^2(\Omega)$ and $L^2_{\mathrm{div}}(\Omega)$, respectively. The proof relies on the Galerkin approximation method. We look for $\left(v^l, \omega^l, b^l\right)$ given as

$$v^l(t,x) = \sum_{i=1}^{n} c_i^l(t) w_i(x), \tag{98}$$

$$b^l(t,x) = \sum_{i=1}^{m} d_i^l(t) z_i(x), \tag{99}$$

$$\omega^l(t,x) = \sum_{i=1}^{l} e_i^l(t) z_i(x) \tag{100}$$

and we require that coefictients $c^l = (c_1^l, \ldots, c_n^l)$, $d^l = (d_1^l, \ldots, d_m^l)$, $e^m = (e_1^l, \ldots, e_l^l)$ solve the following system of ordinary differential equations on $(0,T)$:

$$\left(\partial_t v^l, w_i\right) - \left(G_k\left(\left|v^l\right|^2\right) v^l \otimes v^l, \nabla w_i\right) + \left(T_k\left(\mu^l\right) D(v^l), D(w_i)\right) = 0 \quad \text{for all } i = 1,\ldots,n, \tag{101}$$

$$\left(\partial_t b^l, z_i\right) - \left(b^l v^l, \nabla z_i\right) + \left(T_n(\mu^l)\nabla b^l, \nabla z_i\right) + \left(b_+^l T_m(\omega_+^l) - \frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, z_i\right) = 0 \quad \text{for all } i = 1,\ldots,m, \tag{102}$$

$$\left(\partial_t \omega^l, z_i\right) - \left(\omega^l v^l, \nabla z_i\right) + \left(T_n(\mu^l)\nabla \omega^l, \nabla z_i\right) + \kappa_2\left(T_m(\omega^l)\omega_+^l, z_i\right) = 0 \quad \text{for all } i = 1,\ldots,l, \tag{103}$$

where

$$\mu^l = \frac{b_+^l}{\omega_+^l + \frac{1}{m}} + \frac{1}{n}. \tag{104}$$

We set initial conditions for $(c^l, d^l, e^l)$ given by

$$v^l(0) = \sum_{i=0}^{n}(v_0^n, w_i)w_i, \quad b^l(0) = \sum_{i=0}^{m}(b_0^{m,n,k}, z_i)z_i, \quad \omega^l(0) = \sum_{i=0}^{l}(\omega_0, z_i)z_i. \tag{105}$$

The existence of a solution (101)-(105) follows from Carathodory's theorem. Using estimates established below, a solution can be extended to time interval $[0,T]$.

## 7.1. L-INDEPENDENT ESTIMATES

Multiplying the equation (101) by $c_i^l(t)$ and summing from $i=1$ through $l$, we get

$$\frac{1}{2}\frac{d}{dt}\|v^l\|_2^2 - \left(G_k\left(\left|v^l\right|^2\right)v^l\otimes v^l, \nabla v^l\right) + \left(T_k\left(\mu^l\right)D(v^l), D(v^l)\right) = 0. \tag{106}$$

We see that

$$\left(G_k\left(\left|v^l\right|^2\right)v^l\otimes v^l, \nabla v^l\right) = \frac{1}{2}\left(G_k\left(\left|v^l\right|^2\right)v^l, \nabla\left|v^l\right|^2\right) = \frac{1}{2}\left(v^l, \nabla\Gamma_k\left(\left|v^l\right|^2\right)\right)$$
$$= -\frac{1}{2}\left(\operatorname{div} v^l, \Gamma_k\left(\left|v^l\right|^2\right)\right) = 0.$$

Thus, integrating (106) from 0 to $T$ we have

$$\sup_{t\in(0,T)}\|v^l(t)\|_2^2 + \int_{\Omega^T} T_k(\mu^l)|Dv^l|^2 dxdt \le C(\|v_0\|_2). \tag{107}$$

Using orthonormality of the basis $\{w_i\}$ in $L^2(\Omega)$, we deduce that

$$\sup_{t\in(0,T)}|c^l(t)| \le C(\|v_0\|_2). \tag{108}$$

From the equation (101) and orthonormality of the basis, one can easily deduce

$$|\partial_t c_i^l| \le \left|\left(G_k\left(\left|v^l\right|^2\right)v^l\otimes v^l, \nabla w_i\right)\right| + \left|\left(T_k\left(\mu^l\right)D(v^l), D(w_i)\right)\right|$$
$$\le \frac{9}{2}\left|\left(G_k\left(\left|v^l\right|^2\right)\left|v^l\right|^2, |\nabla w_i|\right)\right| + \sum_{j=1}^{n}\left|c_j^l(t)\right|\left(T_k\left(\mu^l\right)D(w_j), D(w_i)\right).$$

Now, using (29), (32) and the inequality (108), we get

$$|\partial_t c_i^l| \le C(k)\|\nabla w_i\|_2 + C(n,k)\|\nabla w\|_2^2.$$

Using the fact that $\|\nabla w\|_2 \le C(n)$, we get

$$\sup_{t\in(0,T)}|\partial_t c^l(t)| \le C(n,k). \tag{109}$$

Now, multiplying (102) by $d_i^l(t)$ and summing from $i=1$ through $l$, we get

$$\begin{aligned}\left(\partial_t b^l, b^l\right) - \left(b^l v^l, \nabla b^l\right) + \left(T_n(\mu^l)\nabla b^l, \nabla b^l\right) \\ = \left(-b_+^l T_m(\omega_+^l) + \frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, b^l\right).\end{aligned} \tag{110}$$

Using the fact that $\operatorname{div} v^l = 0$ and the definition (29), we get

$$\left(\partial_t b^l, b^l\right) + \frac{1}{n}\left(\nabla b^l, \nabla b^l\right) \le \left(kn, b^l\right). \tag{111}$$

Thus, by Young and Grönwall inequality from (111), we deduce

$$\sup_{t\in(0,T)} \|b^l(t)\|_2^2 + \int_0^T \|\nabla b^l\|_2^2 dt \le C(k,n). \tag{112}$$

Using (112) and orthonormality of the basis in $L^2(\Omega)$, we deduce that

$$\sup_{t\in(0,T)} \left|d^l\right| \le C(k,n). \tag{113}$$

Now, using the equation (102) and orthonormality of the basis in $L^2(\Omega)$, we get

$$\begin{aligned}\left|\partial_t d_i^l\right| &\le \left|\left(b^l v^l, \nabla z_i\right)\right| + \left|\left(T_n(\mu^l)\nabla b^l, \nabla z_i\right)\right| + \left|\left(-b_+^l T_m(\omega_+^l), z_i\right)\right| \\ &\quad + \left|\left(\frac{T_k\left(\mu^l\right)|Dv^l|^2}{1+n^{-1}|Dv^l|^2}, z_i\right)\right| \\ &\le \|b^l\|_2\|v^l\|_2\|\nabla z_i\|_\infty + n\|\nabla b^l\|_2\|\nabla z_i\|_2 + m\|b^l\|_2\|z_i\|_2 \\ &\quad + \left\|T_k\left(\mu^l\right)|Dv^l|^2\right\|_1 \|z_i\|_\infty \\ &\le \|b^l\|_2\|v^l\|_2\|\nabla z_i\|_\infty + n\sup_{t\in(0,T)}\left|d^l\right|\sum_{j=1}^m \|\nabla z_j\|_2\|\nabla z_i\|_2 + m\|b^l\|_2\|z_i\|_2 \\ &\quad + k\sup_{t\in(0,T)}\left|c^l\right|^2 \sum_{j=1}^n \|\nabla w_j\|_2^2\|z_i\|_\infty.\end{aligned}$$

Thus, using (113), (112), (108), (107) we obtain

$$\sup_{t\in(0,T)} \left|\partial_t d^l\right| \le C(k,n,m). \tag{114}$$

Now, multiplying the equation (103) by $e_i^l(t)$ and summing from $i=1$ through $l$, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + \left(\omega^l v^l, \nabla\omega^l\right) + \left(T_n(\mu^l)\nabla\omega^l, \nabla\omega^l\right) = -\kappa_2\left(T_m(\omega^l)\omega_+^l, \omega^l\right). \tag{115}$$

From $\operatorname{div} v^l = 0$ and the fact that the right-hand side is non-negative, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^l\|_2^2 + \frac{1}{n}\left(\nabla\omega^l, \nabla\omega^l\right) \leq 0.$$

Thus, integrating from 0 to T we obtain

$$\sup_{t\in(0,T)} \|\omega^l(t)\|_2^2 + \int_{\Omega^T} \|\nabla\omega^l\|_2^2 dxdt \leq C(n). \tag{116}$$

Now, using (116) and the equation (103), we can deduce that

$$\int_0^T \left\|\partial_t\omega^l\right\|_{W^{-1,2}}^2 dt \leq C(n,m). \tag{117}$$

Indeed, let us consider $\varphi \in W^{1,2}(\Omega)$ such that $\|\varphi\|_{1,2} = 1$. We can write $\varphi$ in the following way:

$$\varphi = \sum_{i=1}^{l} \theta_i z_i + \overline{\varphi},$$

where $(\overline{\varphi}, z_i) = 0$ holds for $i = 1, \ldots, l$.

$$\begin{aligned}
\left|(\partial_t\omega^l, \varphi)\right| &= \left|\left(\partial_t\omega^l, \sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\leq \left|\left(\omega^l v^l, \nabla\sum_{i=1}^{l}\theta_i z_i\right)\right| + \left|\left(T_n(\mu^l)\nabla\omega^l, \nabla\sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\quad + \kappa_2\left|\left(T_m(\omega^l)\omega_+^l, \sum_{i=1}^{l}\theta_i z_i\right)\right| \\
&\leq \|\omega^l\|_2 \left\|v^l\right\|_\infty \|\nabla\sum_{i=1}^{l}\theta_i z_i\|_2 + n\|\nabla\omega^l\|_2\|\nabla\sum_{i=1}^{l}\theta_i z_i\|_2 \\
&\quad + m\kappa_2\|\omega^l\|_2\|\sum_{i=1}^{l}\theta_i z_i\|_2.
\end{aligned}$$

Due to $\|\sum_{i=1}^{l}\theta_i z_i\|_2 \leq \|\varphi\|_2$ and $\|\nabla\sum_{i=1}^{l}\theta_i z_i\|_2 \leq \|\varphi\|_{1,2}$ (which both hold thanks to the Bessel inequality and $\|\sum_{i=1}^{l}\theta_i z_i\|_2^2 + \|\nabla\sum_{i=1}^{l}\theta_i z_i\|_2^2 = \left\|\sum_{i=1}^{l}\theta_i z_i\right\|_{1,2}^2 \leq \|\varphi\|_{1,2}^2$), the Young inequality and (108), we get

$$\left|(\partial_t\omega^l, \varphi)\right|^2 \leq C(n)\|\omega^l\|_2^2\,\|\varphi\|_{1,2}^2 + C(n)\|\nabla\omega^l\|_2^2\,\|\varphi\|_{1,2}^2 + C(m)\|\omega^l\|_2^2\|\varphi\|_2^2.$$

Thus, we get

$$\left\|\partial_t\omega^l\right\|_{-1,2}^2 = \sup_{\varphi\in W^{1,2}(\Omega), \|\varphi\|_{1,2}=1} \left|(\partial_t\omega^l, \varphi)\right|^2 \leq C(n,m)\|\omega^l\|_2^2 + C(n)\|\nabla\omega^l\|_2^2.$$

Finally, using (116) we deduce (117).

## 7.2. TAKING THE LIMIT $L \to \infty$

Using estimates (108), (109) and (113), (114), we can find a subsequence (which we do not relabel) such that

$$c^l \rightharpoonup^* c \quad \text{weakly* in } W^{1,\infty}(0,T)^n,$$
$$d^l \rightharpoonup^* d \quad \text{weakly* in } W^{1,\infty}(0,T)^m.$$

Using the Arzela-Ascoli theorem and estimates (109) and (113), we conclude that

$$c^l \to c \qquad \text{strongly in } C(0,T)^n,$$
$$d^l \to d \qquad \text{strongly in } C(0,T)^m.$$

Based on definitions (98), (99) and the above convergences, we deduce the existence of a sequence such that

$$v^l \to v = \sum_{i=1}^{n} c_i w_i \qquad \text{strongly in } C(0,T,W^{1,2}_{\mathrm{div}}(\Omega)), \tag{118}$$

$$b^l \to b = \sum_{i=1}^{m} d_i z_i \qquad \text{strongly in } C(0,T,W^{1,2}(\Omega)).$$

Using (116) and (117) and the Aubin-Lions lemma, we get

$$\omega^l \rightharpoonup^* \omega \qquad \text{weakly* in } L^2\left(0,T;W^{1,2}\right) \cap L^\infty\left(0,T;L^2(\Omega)\right), \tag{119}$$
$$\partial_t \omega^l \rightharpoonup \partial_t \omega \qquad \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right), \tag{120}$$
$$\omega^l \to \omega \qquad \text{strongly in } L^2\left(0,T;L^2(\Omega)\right). \tag{121}$$

Having the above estimates, it is easy to identify the limit of the system (98)-(105) to get

$$(\partial_t v, w_i) - \left(G_k\left(|v|^2\right) v \otimes v, \nabla w_i\right) + \left(T_k\left(\widetilde{\mu}^{n,m}\right) D(v), D(w_i)\right) = 0 \qquad \text{for all } i = 1,\dots,n, \tag{122}$$

$$(\partial_t b, z_i) - (bv, \nabla z_i) + (T_n(\widetilde{\mu}^{n,m})\nabla b, \nabla z_i) + (-b_+ T_m(\omega_+), z_i) = \left(\frac{T_k\left(\widetilde{\mu}^{n,m}\right)|Dv|^2}{1+n^{-1}|Dv|^2}, z_i\right) \qquad \text{for all } i = 1,\dots,m, \tag{123}$$

$$\langle \partial_t \omega, z\rangle - (\omega v, \nabla z) + (T_n(\widetilde{\mu}^{n,m})\nabla \omega, \nabla z) = -\kappa_2\left(T_m(\omega)\omega_+, z\right) \qquad \text{for all } z \in W^{1,2}(\Omega), \tag{124}$$

where

$$\widetilde{\mu}^{n,m} = \frac{b_+}{\omega_+ + \frac{1}{m}} + \frac{1}{n}.$$

To obtain the system (91)-(93), we show the bounds for $\omega$. This will allow us to replace $\widetilde{\mu}^{n,m}$ with $\mu^{n,m}$ in equations (122)-(124). Additionally, we will be able to replace $T_m(\omega_+)$ and $T_m(\omega)$ with $\omega$.

## 7.3. MINIMUM AND MAXIMUM PRINCIPLE FOR $\omega$

We apply $\omega_-$ as a test function in (124) and obtain

$$\langle \partial_t \omega, \omega_- \rangle - (\omega v, \nabla \omega_-) + (T_n(\widetilde{\mu}^{n,m}) \nabla \omega, \nabla \omega_-) = -\kappa_2 (T_m(\omega)\omega_+, \omega_-).$$

We see that right hand side of the above equality is equal to zero and thus

$$\frac{1}{2}\frac{d}{dt}\|\omega_-\|_2^2 - (\omega_- v, \nabla \omega_-) + (T_n(\widetilde{\mu}^{n,m}) \nabla \omega_-, \nabla \omega_-) = 0.$$

Using the fact that $\operatorname{div} v = 0$, we conclude that, second term is equal to zero. Additionally, using nonnegativity of the second term, we finally get

$$\frac{d}{dt}\|\omega_-\|_2^2 \leq 0 \ \text{ so } \ \forall t \in (0,T) \ \ \|\omega_-(t,\cdot)\|_2^2 = 0. \tag{125}$$

Thus, using (125), we conclude that $\omega \geq 0$ almost everywhere in $\Omega^T$. From this, we see that $\widetilde{\mu}^{n,m} = \mu^{n,m}$, and thus we can rewrite (124) in the following way:

$$\begin{aligned} \langle \partial_t \omega, z\rangle - (\omega v, \nabla z) + (T_n(\mu^{n,m})\nabla \omega, \nabla z) = -\kappa_2 (T_m(\omega)\omega, z) \\ \text{for all } z \in W^{1,2}(\Omega), \end{aligned} \tag{126}$$

Similarly, from (122)-(123) we get (91)-(92). Now, we will show the upper bound on $\omega$. Let us first test the equation (126) using $(\omega - \omega_{\max})_+$:

$$\begin{aligned} \left\langle \partial_t \omega, (\omega - \omega_{\max})_+ \right\rangle - \left(\omega v, \nabla (\omega - \omega_{\max})_+\right) \\ + \left(T_n(\mu^{n,m}) \nabla \omega, \nabla (\omega - \omega_{\max})_+\right) \\ = -\kappa_2 \left(T_m(\omega)\omega, (\omega - \omega_{\max})_+\right). \end{aligned}$$

From this, using the fact that $\operatorname{div} v = 0$, we have

$$\begin{aligned} \left\langle \partial_t (\omega - \omega_{\max})_+, (\omega - \omega_{\max})_+ \right\rangle + \left(\nabla (\omega - \omega_{\max})_+ v, (\omega - \omega_{\max})_+\right) \\ + \left(T_n(\mu^{n,m}) \nabla (\omega - \omega_{\max})_+, \nabla (\omega - \omega_{\max})_+\right) \\ = -\kappa_2 \left(T_m(\omega)\omega, (\omega - \omega_{\max})_+\right). \end{aligned}$$

Since $\operatorname{div} v = 0$, this gives us the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(\omega - \omega_{\max})_+\|_2^2 \leq -\kappa_2 \left(T_m(\omega)\omega, (\omega - \omega_{\max})_+\right) \leq 0 \\ \text{so } \ \forall t \in (0,T) \ \ \|(\omega(t,\cdot) - \omega_{\max})_+\|_2 = 0. \end{aligned}$$

Thus, $\omega \leq \omega_{\max}$ almost everywhere in $\Omega^T$. Let us recall that $m \geq \omega_{\max}$ and thus from (126) we get

$$\langle \partial_t \omega, z\rangle - (\omega v, \nabla z) + (T_n(\mu^{n,m})\nabla \omega, \nabla z) = -\kappa_2 \left(\omega^2, z\right) \quad \text{for all } z \in W^{1,2}(\Omega), \tag{127}$$

which is exactly equal to (93). Now, let us test the equation (127) using $\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+$:

$$\left\langle \partial_t \omega, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+ \right\rangle - \left(\omega v, \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) + \left(T_n\left(\mu^{n,m}\right)\nabla\omega, \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) = -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right).$$

Using the fact that $\nabla\omega = \nabla\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)$, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\|_2^2 - \left(\frac{\kappa_2\omega_{\max}^2}{(1+\kappa_2\omega_{\max}t)^2}, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right) \le -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\|_2^2 &\le -\kappa_2\left(\omega^2 - \frac{\omega_{\max}^2}{(1+\kappa_2\omega_{\max}t)^2}, \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+\right)\\ &= -\kappa_2\left(\left(\omega + \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right), \left(\omega - \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\right)_+^2\right)\\ &\le 0.\end{aligned}$$

Thus, by integration and using $\omega_0 \le \omega_{\max}$, we conclude that $\omega \le \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}$ almost everywhere in $\Omega^T$. Now, we will obtain the bound from below by testing the equation (127) by $\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-$:

$$\left\langle \partial_t \omega, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_- \right\rangle - \left(\omega v, \nabla\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right) + \left(T_n\left(\mu^{n,m}\right)\nabla\omega, \nabla\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right) = -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right).$$

Again, we deduce that the second term is equal to zero and the third one is positive:

$$\frac{1}{2}\frac{d}{dt}\|\left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\|_2^2 - \left(\frac{\kappa_2\omega_{\min}^2}{(1+\kappa_2\omega_{\min}t)^2}, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right) \le -\kappa_2\left(\omega^2, \left(\omega - \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\|_2^2 &\leq -\kappa_2\left(\omega^2-\frac{\omega_{\min}^2}{(1+\kappa_2\omega_{\min}t)^2},\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-\right)\\ &\leq -\kappa_2\left(\omega+\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t},\left(\omega-\frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}\right)_-^2\right)\\ &\leq 0.\end{aligned}$$

By integration and the fact that $\omega_0 \geq \omega_{\min}$, we conclude that $\omega \geq \frac{\omega_{\min}}{1+\kappa_2\omega_{\min}t}$ almost everywhere in $\Omega^T$.
For now on we will refrain from showing the attainment of initial data. The methodology will be shown for a more complex case, that is, in the proof of Theorem 1. □

# 8. PROOF OF THEOREM 8

We use $(v^m, \omega^m, b^m)$ to denote a solution of the system (81)-(85), whose existence was established in Theorem 11. Our goal is to let $m \to \infty$ and thus prove Theorem 10.

## 8.1. M-INDEPENDENT ESTIMATES

Repeating the procedure from (106)-(107), we deduce that

$$\sup_{t\in(0,T)}\|v^m(t)\|_2^2+\int_{\Omega^T}T_k(\mu^{n,m})|Dv^m|^2dxdt \leq C, \tag{128}$$

$$\sup_{t\in(0,T)}|c^m(t)|\leq C, \qquad \sup_{t\in(0,T)}|\partial_t c^m(t)|\leq C(n,k). \tag{129}$$

We multiply (92) by $d_i^m(t)$ and sum from $i=1$ through $m$ to obtain

$$\begin{aligned}(\partial_t b^m, b^m)-(b^m v^m,\nabla b^m)+(T_n(\mu^{n,m})\nabla b^m,\nabla b^m)\\ =\left(-b_+^m\omega^m+\frac{T_k(\mu^{n,m})|Dv^m|^2}{1+n^{-1}|Dv^m|^2},b^m\right).\end{aligned}$$

Using (29), (94), (90), the fact that $\frac{n^{-1}|Dv^m|^2}{1+n^{-1}|Dv^m|^2}\leq 1$ and $\operatorname{div} v^m=0$, we get

$$(\partial_t b^m, b^m)+\frac{1}{n}(\nabla b^m,\nabla b^m)\leq (kn,|b^m|).$$

Using the Young inequality and Grönwall lemma, we get

$$\sup_{t\in(0,T)} \|b^m(t)\|_2^2 + \frac{1}{n}\int_{\Omega^T} \|\nabla b^m\|_2^2 dt \le C(n,k). \tag{130}$$

Using the equation (92) and (130), (128), (94), we conclude (in the same way as in (117)-(118)) that

$$\int_0^T \|\partial_t b^m\|_{W^{-1,2}(\Omega)}^2 \, dt \le C(n,k). \tag{131}$$

Testing the equation (93) by $\omega^m$, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^m\|_2^2 + (\omega^m v^m, \nabla\omega^m) + (T_n(\mu^{n,m})\nabla\omega^m, \nabla\omega^m) = -\kappa_2\left((\omega^m)^2, \omega^m\right).$$

Using the fact that $\operatorname{div} v^m = 0$ and (90), (94) we get

$$\frac{1}{2}\frac{d}{dt}\|\omega^m\|_2^2 + \frac{1}{n}\|\nabla\omega^m\|_2^2 \le 0,$$

and thus

$$\int_0^T \|\omega^m\|_{1,2}^2 \, dt \le C(n,k). \tag{132}$$

Using the equation (93) and estimates (128), (132), one can deduce (again, in the same way as in (117)-(118)) that

$$\int_0^T \|\partial_t \omega^m\|_{W^{-1,2}(\Omega)} \, dt \le C(n,k). \tag{133}$$

## 8.2. TAKING THE LIMIT $M \to \infty$

By the estimate (128) we can find a subsequence (which we do not relabel) such that

$$c^m \rightharpoonup^* c \quad \text{weakly* in } W^{1,\infty}(0,T)^n.$$

Using the Arzela-Ascoli theorem and the estimate (129), we conclude that

$$c^m \to c \qquad \text{strongly in } C(0,T)^n. \tag{134}$$

Based on the definition (95) and the convergence (134), one can deduce

$$v^m \to v = \sum_{i=1}^n c_i w_i \qquad \text{strongly in } C(0,T,W^{1,2}_{\operatorname{div}}(\Omega)).$$

Using (132), (133) and (90) and the Aubin-Lions lemma, we find a subsequence such that

$$\begin{aligned}
\omega^m &\rightharpoonup \omega && \text{weakly in } L^2\left(0,T;W^{1,2}(\Omega)\right),\\
\partial_t\omega^l &\rightharpoonup \partial_t\omega && \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right),\\
\omega^m &\rightharpoonup^* \omega && \text{weakly* in } L^\infty\left(0,T;L^\infty(\Omega)\right),\\
\omega^m &\to \omega && \text{strongly in } L^2\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Using (130), (131) and the Aubin-Lions lemma, we extract a subsequence such that

$$\begin{aligned}
b^m &\rightharpoonup^* b && \text{weakly* in } L^2\left(0,T;W^{1,2}(\Omega)\right)\cap L^\infty\left(0,T;L^2(\Omega)\right),\\
\partial_t b^l &\rightharpoonup \partial_t b && \text{weakly in } L^2\left(0,T;W^{-1,2}(\Omega)\right),\\
b^m &\to b && \text{strongly in } L^2\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Having the above estimates, it is easy to identify the limit of the system (91) - (95) to obtain

$$(v_{,t},w_i)-\left(G_k\left(|v|^2\right)v\otimes v,\nabla w_i\right)+\left(T_k\left(\widetilde{\mu^n}\right)D(v),D(w_i)\right)=0 \quad \text{for all } i=1,\dots,n, \tag{135}$$

$$\langle b_{,t},z\rangle-(bv,\nabla z)+\left(T_n(\widetilde{\mu^n})\nabla b,\nabla z\right)+\left(b_+\omega-\frac{T_k\left(\widetilde{\mu^n}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2},z\right)=0 \quad \forall z\in W^{1,2}(\Omega)\text{a.a. } t\in(0,T), \tag{136}$$

$$\langle \omega_{,t},z\rangle-(\omega v,\nabla z)+\left(T_n(\widetilde{\mu^n})\nabla\omega,\nabla z\right)+\kappa_2\left(\omega^2,z\right)=0 \quad \forall z\in W^{1,2}(\Omega)\text{a.a. } t\in(0,T), \tag{137}$$

where

$$\widetilde{\mu^n}=\frac{b_+}{\omega}+\frac{1}{n}. \tag{138}$$

Now, we will show bounds for $b$ from which we will conclude that $\widetilde{\mu^n}=\mu^n$. By doing so, we will show the existence of a solution to (75) - (78).

## 8.3. MINIMUM PRINCIPLE FOR B

Firstly, let us test the equation (136) with $z=b_-$. We get

$$\begin{aligned}
\langle b_{,t},b_-\rangle-(bv,\nabla b_-)&+\left(T_n(\widetilde{\mu^n})\nabla b,\nabla b_-\right)\\
&=(-b_+\omega,b_-)+\left(\frac{T_k\left(\widetilde{\mu^n}\right)|D(v)|^2}{1+n^{-1}|D(v)|^2},b_-\right).
\end{aligned}$$

Using the fact that $\operatorname{div} v = 0$ to the second term of left hand side, positivity of the third term on left hand side and non-positivity of the second term on right hand side, we get

$$\frac{1}{2}\frac{d}{dt}\|b_-\|_2^2 \le (-b_+\omega+, b_-) = 0.$$

Thus, we deduce that $b \ge 0$ almost everywhere in $\Omega^T$. From this, it follows that $\widetilde{\mu^n} = \mu^n$ and that, the positive part of $b$ in (136) can be dropped. Thus, the existence of a solution to the system (75) - (78) is established. Now, let us test the equation (76) with $z = \left(b - \frac{1}{k(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\right)_-$. Again, using $\operatorname{div} v = 0$ and positivity of third term on left hand side and negativity of second term on right hand side, we get

$$\left\langle b_{,t}, \left(b - \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}\right)_- \right\rangle \le \left(-b\omega, \left(b - \frac{b_{\min}}{(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}\right)_-\right).$$

The inequality can be rewritten in the following way:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\| \left(b - \frac{1}{k\,(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\right)_- \right\|_2^2 \\
&\le \left(-b\omega + \frac{\omega_{\max}}{k\,(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}+1}}, \left(b - \frac{1}{k\,(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\right)_-\right). \\
&\le \frac{\omega_{\max}}{1+\kappa_2\omega_{\max}t}\left(-b + \frac{1}{k\,(1+\kappa_2\omega_{\max}t)^{\frac{1}{\kappa_2}}}, \left(b - \frac{1}{k\,(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}\right)_-\right) \\
&\le 0.
\end{aligned}$$

Using integration from 0 to $t$ and using the fact that $b_0^{n,k} \ge \frac{1}{k}$, we conclude that $b \ge \frac{1}{k(1+\kappa_2\omega_{\min}t)^{\frac{1}{\kappa_2}}}$ almost everywhere in $\Omega^T$, and thus we prove (74).

## 8.4. REMAINING INEQUALITY

Now, we will establish (80). Let us test the equation (76) with $\frac{\varphi}{2\sqrt{b}}$, where $\varphi \in \mathcal{D}(\Omega)$ and $\varphi \ge 0$. Thus, we get

$$\begin{aligned}
\langle b_{,t}, \frac{\varphi}{2\sqrt{b}}\rangle - \left(bv, \nabla\frac{\varphi}{2\sqrt{b}}\right) + \left(T_n(\mu^n)\nabla b, \nabla\frac{\varphi}{2\sqrt{b}}\right)& \\
= \left(-b\omega, \frac{\varphi}{2\sqrt{b}}\right) + \left(\frac{T_k(\mu^n)\,|D(v)|^2}{1+n^{-1}|D(v)|^2}, \frac{\varphi}{2\sqrt{b}}\right)&.
\end{aligned}$$

Using non-negativity of the last term of the right hand side and $\operatorname{div} v = 0$, we get

$$\left\langle \left(\sqrt{b}\right)_t, \varphi \right\rangle + \left(\nabla b v, \frac{\varphi}{2\sqrt{b}}\right) + \left(T_n(\mu^n)\nabla b, \frac{\nabla\varphi}{2\sqrt{b}}\right) - \left(T_n(\mu^n)\nabla b, \frac{\varphi}{4b^{\frac{3}{2}}}\nabla b\right) \\ \geq -\frac{1}{2}\left(\sqrt{b}\omega, \varphi\right).$$

Let us observe that the last term of the left hand side is not positive. Thus, we get

$$\left\langle \left(\sqrt{b}\right)_t, \varphi \right\rangle + \left(\nabla\left(\sqrt{b}\right) v, \varphi\right) + \left(T_n(\mu^n)\nabla\sqrt{b}, \nabla\varphi\right) \geq -\frac{1}{2}\left(\sqrt{b}\omega, \varphi\right).$$

After integrating from 0 to $t$, we get

$$\left(\sqrt{b}(t), \varphi\right) - \int_0^t \left(\sqrt{b}v, \nabla\varphi\right) d\tau + \int_0^t \left(T_n(\mu^n)\nabla\sqrt{b}, \nabla\varphi\right) d\tau \\ \geq \left(\sqrt{b_0^{n,k}}, \varphi\right) - \frac{1}{2}\int_0^t \left(\sqrt{b}\omega, \varphi\right) d\tau.$$

# 9. PROOF OF THEOREM 7

Let $(v^n, b^n, \omega^n)$ be a solution to the problem (75)-(78), whose existence is guaranteed by Theorem 10. Our goal is to pass with $n \to \infty$ in (75)-(78) to obtain (54)-(56), and thus to prove Theorem 9.

## 9.1. N-INDEPENDENT ESTIMATES

Proceeding as before, we get

$$\|v^n(t)\|_2^2 + 2\int_0^t \int_\Omega T_k(\mu^n)|D(v^n)|^2 dx d\tau = \|v_0^n\|_2^2. \tag{139}$$

Thus, we deduce the following estimate:

$$\sup_{t\in(0,T)} \|v^n(t)\|_2^2 + \int_{\Omega^T} T_k(\mu^n)|D(v^n)|^2 dx d\tau \leq C. \tag{140}$$

Now, using bounds on $\omega$ (73) and $b$ (74) and Korn inequality, we deduce

$$\int_0^T \|v^n\|_{1,2}^2 dx d\tau \leq C(k). \tag{141}$$

Using the equation (75) and the inequality (141), one can deduce

$$\int_0^T \|\partial_t v^n\|_{-1,2}^2 \, dx d\tau \leq C(k). \tag{142}$$

Using standard interpolation inequality $\|u\|_{\frac{10}{3}} \leq C \|u\|_2^{\frac{2}{5}} \|u\|_{1,2}^{\frac{3}{5}}$ and (140), (141), we get

$$\int_0^T \|v^n\|_{\frac{10}{3}}^{\frac{10}{3}} \, d\tau \leq C(k). \tag{143}$$

Now, we will focus on uniform estimates for $b^n$. First, for arbitrary $a > 0$ we set $z = T_a(b^n)$ in (76) and obtain

$$\begin{aligned} \langle b^n_{,t}, T_a(b^n)\rangle - (b^n v^n, \nabla T_a(b^n)) + (T_n(\mu^n)\nabla b^n, \nabla T_a(b^n)) \\ = \left( -b^n \omega^n + \frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, T_a(b^n) \right). \end{aligned}$$

Using the definition (30), we get

$$\langle \partial_t b^n, T_a(b^n)\rangle = \frac{d}{dt} \|\Theta_a(b^n)\|_1 \,. \tag{144}$$

From the inequality (140), we obtain

$$\int_0^T \left( \frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, T_a(b^n) \right) \leq aC. \tag{145}$$

Using the fact that $\operatorname{div} v^n = 0$, we get

$$(b^n v^n, \nabla T_a(b^n)) = -(\nabla b^n v^n, T_a(b^n)) = -(v^n, \nabla \Theta_a(b^n)) = (\operatorname{div} v^n, \Theta_a(b^n)) = 0.$$

We also have

$$(T_n(\mu^n)\nabla b^n, \nabla T_a(b^n)) = \int_\Omega T_n(\mu^n)\,|\nabla T_a(b^n)|^2 \, dx. \tag{146}$$

Combining (144), (145) and (146), we get

$$\frac{d}{dt} \|\Theta_a(b^n)\|_1 + \int_\Omega T_n(\mu^n)\,|\nabla T_a(b^n)|^2 \, dx \leq Ca.$$

By integration of the above inequality from 0 to $T$, we get

$$\sup_{t\in(0,T)} \|\Theta_a(b^n(t))\|_1 + \int_{\Omega^T} T_n(\mu^n)\,|\nabla T_a(b^n)|^2 \, dx d\tau \leq CaT + 2\,\|\Theta_a(b^n(0))\|_1 \,. \tag{147}$$

Using (30) one can show that

$$\text{if } \; b^n \geq a \quad \text{then} \quad \Theta_a(b^n) \geq \frac{1}{2} a b^n$$

and

$$\text{if } b^n < a \quad \text{then} \quad \Theta_a(b^n) = \frac{1}{2}(b^n)^2.$$

Thus, we get

$$\begin{aligned} \|b^n(t)\|_1 &\leq \int_{b^n(t)\geq a} \frac{2}{a}\Theta_a(b^n(t))dx + \int_{b^n(t)<a} \sqrt{2\Theta_a(b^n(t))}dx \\ &\leq C(a)\|\Theta_a(b^n(t))\|_1 + C(a). \end{aligned} \tag{148}$$

Using (147), (148) and (60), the following inequality can be deduced:

$$\sup_{t\in(0,T)} \|b^n(t)\|_1 + \int_{\Omega^T} T_n(\mu^n)\,|\nabla T_a(b^n)|^2\,dxd\tau \leq C(a). \tag{149}$$

Now, we test the equation (76) using $z = \frac{1}{(b^n)^\lambda}$ (which is a viable test function based on (74)), where $\lambda \in (0,1)$:

$$\frac{d}{dt}\int_\Omega \frac{(b^n)^{1-\lambda}}{1-\lambda}dx - \lambda \int_\Omega T_n(\mu^n)\frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}}dx = \left(-b^n\omega^n + \frac{T_k(\mu)\,|D(v)|^2}{1+n^{-1}|D(v)|^2}, \frac{1}{(b^n)^\lambda}\right).$$

Thus, integrating from 0 to $t$ and taking supremum over $t \in (0,T)$, we obtain the following inequality:

$$\lambda\int_{\Omega^T} T_n(\mu^n)\frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}}dxdt \leq \int_0^T \left(b^n\omega^n, \frac{1}{(b^n)^\lambda}\right)dt + \frac{1}{1-\lambda}\sup_{t\in(0,T)}\int_\Omega (b^n)^{1-\lambda}\,dx.$$

Using (149) and (73), we can bound right hand side uniformly and finally obtain

$$\int_{\Omega^T} \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}}|\nabla b^n|^2 dxdt \leq C(\lambda^{-1}) \quad \forall\ \lambda \in (0,1). \tag{150}$$

Now, we set $z = \frac{1}{b^n}$ in (76), and using the fact that $\operatorname{div} v^n = 0$, we obtain

$$\frac{d}{dt}\int_\Omega \ln b^n(t)dx - \int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dx = \left(-b^n\omega^n + \frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, \frac{1}{b^n}\right).$$

After integrating from 0 to $t$, we deduce

$$\begin{aligned} -\int_{b^n<1} \ln b^n(t)dx &+ \int_0^t\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dxdt + \int_0^t \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, \frac{1}{b^n}\right)dt \\ &\leq \int_0^t (\omega^n, 1) + \|\ln b_0^{n,k}\|_1. \end{aligned}$$

With the proper usage of supremum, we obtain

$$\begin{aligned} \sup_{t\in(0,T)} -\int_{b^n<1} \ln b^n(t)dx &+ \int_0^T\int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dxdt + \int_0^T \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, \frac{1}{b^n}\right)dt \\ &\leq 2\int_0^T (\omega^n, 1)\,dt + 2\|\ln b_0^{n,k}\|_1. \end{aligned}$$

Using the fact that

$$\|\ln b^n(t)\|_1 = -\int_{\Omega\cap\{b^n(t)<1\}} \ln b^n(t)dx + \int_{\Omega\cap\{b^n(t)>1\}} \ln b^n(t)dx,$$

we obtain

$$\sup_{t\in(0,T)} \|\ln b^n(t)\|_1 + \int_0^T \int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dxdt + \int_0^T \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, \frac{1}{b^n}\right) dt$$
$$\leq 2\int_0^T (\omega^n, 1)\, dt + \sup_{t\in(0,T)} \|b^n(t)\|_1 + 2\|\ln b_0^{n,k}\|_1.$$

Thus, by (73), (149) (taken with $a=1$), (11), (60), (61), we get

$$\sup_{t\in(0,T)} \|\ln b^n(t)\|_1 + \int_0^T \int_\Omega \frac{T_n(\mu^n)}{(b^n)^2}|\nabla b^n|^2 dxdt + \int_0^T \left(\frac{T_k(\mu^n)\,|D(v)|^2}{1+n^{-1}|D(v)|^2}, \frac{1}{b^n}\right) dt \leq C. \quad (151)$$

Combining (151) with (150), (61), (60), we get

$$\sup_{t\in(0,T)} \|\ln b^n(t)\|_1 + \int_0^T \int_\Omega \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}}|\nabla b^n|^2 dxdt + \int_0^T \int_\Omega \frac{T_k(\mu^n)\,|D(v)|^2}{b^n\,(1+n^{-1}|D(v)|^2)} dxdt$$
$$\leq C(\lambda^{-1}) \quad \forall\ \lambda\in(0,1]. \quad (152)$$

We see that, based on (140), (29), for all $k\in\mathbb{N}$

$$k\int_{\Omega^T\cap\{\mu^n\geq k\}} \frac{|Dv^n|^2}{1+n^{-1}|Dv^n|^2} dxdt \leq C.$$

Additionally, based on (152) with some specific $\lambda$ i.e. $\lambda=\frac{1}{2}$, (29), (73), (66), we have that for all $k\in\mathbb{N}$

$$\int_{\Omega^T\cap\{\mu^n\leq k\}} \frac{|Dv^n|^2}{1+n^{-1}|Dv^n|^2} dxdt \leq C.$$

Thus, we get

$$\int_0^T \int_\Omega \frac{|Dv^n|^2}{1+n^{-1}|Dv^n|^2} dxdt \leq C, \quad (153)$$

where $C$ does not depend on $k$ and $n$. Next, we will focus on estimates on $T_n(\mu^n)$ that are uniform with respect to n and k . Using the definition (29), we get

$$T_n\left(\frac{b^n}{\omega^n}\right) \leq T_n(\mu^n) \leq T_n\left(\frac{b^n}{\omega^n}\right) + \frac{1}{n}$$

and thus

$$\min\left\{1, \frac{1}{\omega^n}\right\} T_n(b^n) \leq T_n(\mu^n) \leq \max\left\{1, \frac{1}{\omega^n}\right\} T_n(b^n) + \frac{1}{n}.$$

Due to (73), we finally get

$$C_1 T_n(b^n) \le T_n(\mu^n) \le C_2 T_n(b^n) + \frac{1}{n}. \tag{154}$$

Thus, by (152)

$$\begin{aligned}
&\int_0^T \|\nabla\left[(T_n(b^n))^{1-\frac{\lambda}{2}}\right]\|_2^2 dt = C(\lambda)\int_{\Omega^T} \frac{|\nabla T_n(b^n)|^2}{(T_n(b^n))^{\lambda}} dxdt \\
&= C(\lambda)\int_{\Omega^T\cap\{b^n\le n\}} \frac{|\nabla T_n(b^n)|^2}{(b^n)^{\lambda}} dxdt = C(\lambda)\int_{\Omega^T\cap\{b^n\le n\}} \frac{T_n(b^n)}{(b^n)^{1+\lambda}} |\nabla T_n(b^n)|^2 dxdt \\
&\le C(\lambda)\int_{\Omega^T\cap\{b^n\le n\}} \frac{T_n(\mu^n)}{(b^n)^{1+\lambda}} |\nabla T_n(b^n)|^2 dxdt \le C(\lambda^{-1}).
\end{aligned} \tag{155}$$

Combining (149) (i.e. with $a = 1$) with (155), we get

$$\sup_{t\in(0,T)} \|(T_n(b^n))^{1-\lambda/2}\|_1 + \int_0^T \left\|\nabla\left[(T_n(b^n))^{1-\frac{\lambda}{2}}\right]\right\|_2^2 dt \le C(\lambda^{-1}). \tag{156}$$

Using interpolation inequality $\|f\|_{\frac{8}{3}}^{\frac{8}{3}} \le C\|f\|_1^{\frac{2}{3}} \|f\|_{1,2}^2$ and inequality (156), we get

$$\int_{\Omega^T} \left\|(T_n(\mu^n))^{1-\lambda/2}\right\|_{\frac{8}{3}}^{\frac{8}{3}} \le C(\lambda^{-1}),$$

and thus

$$\int_{\Omega^T} \|T_n(\mu^n)\|_{\frac{8-4\lambda}{3}}^{\frac{8-4\lambda}{3}} \le C(\lambda^{-1}) \text{ for all } \lambda \in (0,1). \tag{157}$$

Also, let us observe that (157) implies

$$\int_{\Omega^T} \|(T_n(\mu^n))^{\alpha}\|_q^q \le C(q) \text{ for all } q \in \left[1, \frac{8}{3\alpha}\right) \text{ and } \alpha \in (0,1]. \tag{158}$$

Now, we continue with k-dependent estimates that will be useful to obtain $n \to \infty$ limit. From (152), maximum principle for $\omega^n$ (73) and minimum principle for $b^n$ (74), we get

$$\int_{\Omega^T} \frac{|\nabla b^n|^2}{(b^n)^{1+\lambda}} dxdt \le C(\lambda^{-1}, k)$$

which combined with (149), yields

$$\int_0^T \left\|(b^n)^{\frac{1-\lambda}{2}}\right\|_{1,2}^2 dxdt \le C(\lambda^{-1}, k).$$

Using the $W^{1,2} \subset L^6$ embedding, we get

$$\int_0^T \left\|(b^n)^{1-\lambda}\right\|_3 dxdt \le C(\lambda^{-1}, k). \tag{159}$$

Finally, using the interpolation inequality

$$\left\|(b^n)^{1-\lambda}\right\|_{\frac{5}{3}}^{\frac{5}{3}} \leq C\|(b^n)^{1-\lambda}\|_1^{\frac{2}{3}} \left\|(b^n)^{1-\lambda}\right\|_3$$

and (149), (159), we conclude that

$$\int_0^T \|b^n\|_{\frac{5-\lambda}{3}}^{\frac{5-\lambda}{3}} dxdt \leq C(\lambda^{-1}, k). \tag{160}$$

Now, we proceed with k dependent estimates on the diffusion term. For any $q \in (1, \frac{5}{4})$, due to (160) and (152), we get

$$\begin{aligned}
\int_{\Omega^T} \left|\sqrt{T_n(\mu^n)}\nabla b^n\right|^q dxdt &= \int_{\Omega^T} \left(\frac{T_n(\mu^n)\,|\nabla b^n|^2}{(b^n)^{1+\lambda}}\right)^{q/2} (b^n)^{\frac{(1+\lambda)q}{2}} dxdt \\
&\leq \left(\int_{\Omega^T} \frac{T_n(\mu^n)\,|\nabla b^n|^2}{(b^n)^{1+\lambda}} dxdt\right)^{q/2} \left(\int_{\Omega^T} (b^n)^{\frac{(1+\lambda)q}{2-q}} dxdt\right)^{\frac{2-q}{2}} \\
&\leq C(\lambda^{-1}, k),
\end{aligned}$$

where $\lambda < \frac{5}{3}\frac{2-q}{q} - 1$, which implies that $\frac{(1+\lambda)q}{2-q} < \frac{5}{3}$. Therefore, we have the following estimate:

$$\int_{\Omega^T} \left|\sqrt{T_n(\mu^n)}\nabla b^n\right|^{\frac{5-\lambda}{4}} dxdt \leq C(\lambda^{-1}, k). \tag{161}$$

Thus, combining the above inequality with (66), (73), (74) and (160), we get

$$\int_{\Omega^T} \|b^n\|_{1,\frac{5-\lambda}{4}}^{\frac{5-\lambda}{4}} dxdt \leq C(\lambda^{-1}, k). \tag{162}$$

Notice that from $\frac{79}{80-\lambda} > \frac{79}{80} = \frac{4}{5} + \frac{3}{16}$, the Hölder inequality, (161) and (157), it follows that

$$\int_{\Omega^T} |T_n(\mu^n)\nabla b^n|^{\frac{80-\lambda}{79}} dxdt \leq C(\lambda^{-1}, k). \tag{163}$$

Additionally, we can observe that due to (163), (154), (53),

$$\int_{\Omega^T} \left|(T_n(\mu^n))^{\alpha}\,\nabla b^n\right|^{\frac{80-\lambda}{79}} dxdt \leq C(\lambda^{-1}, k) \text{ for all } \alpha \in [0,1]. \tag{164}$$

In order to obtain uniform bound on $\partial_t b^n$, it remains to estimate convective term $b^n v^n$. Let us observe that

$$\|b^n v^n\|_{\frac{10-\lambda}{9}} \leq \|v^n\|_{\frac{10}{3}} \|b^n\|_{\frac{10}{3}\frac{10-\lambda}{20+\lambda}}, \tag{165}$$

where $\frac{10}{3}\frac{10-\lambda}{20+\lambda} \in (10/7, 5/3)$, and thus $\frac{10}{3}\frac{10-\lambda}{20+\lambda} < \frac{5-\lambda^*}{3}$ for some $\lambda^* \in (0,1)$. Thus, we can conclude that based on (160), (143) and (165) we have

$$\|b^n v^n\|_{\frac{10-\lambda}{9}} \leq C(k, \lambda^{-1}). \tag{166}$$

Using the equation (76) tested with $z \in W^{1,\frac{80-\lambda}{1-\lambda}}(\Omega)$, combined with inequalities (166), (163), (73), (160), (140), we can conclude with the help of the following inclusion:
$W^{1,\frac{80-\lambda}{1-\lambda}}(\Omega) \subset W^{1,80}(\Omega) \subset C^{0,\frac{77}{80}}(\Omega)$ that

$$\int_0^T \|\partial_t b^n\|_{-1,\frac{80-\lambda}{79}} \leq C(k,\lambda^{-1}). \tag{167}$$

Finally, we derive k-independent estimates for $\omega^n$. We set testing function $z = \omega^n$ in (77) and obtain

$$\frac{1}{2}\frac{d}{dt}\|\omega^n\|_2^2 + \int_\Omega T_n(\mu^n)\,|\nabla\omega^n|^2 = -\kappa_2\left((\omega^n)^2, \omega^n\right).$$

Thus, integrating from 0 to T we get

$$\int_{\Omega^T} T_n(\mu^n)\,|\nabla\omega^n|^2\,dxdt \leq C, \tag{168}$$

which, after using (73) and (74), implies that

$$\int_0^T \|\omega^n\|_{1,2}^2\,dxdt \leq C(k). \tag{169}$$

We see that due to the Hölder inequality,

$$\begin{aligned} \|T_n(\mu^n)\,\nabla\omega^n\|_{\frac{16-\lambda}{11}} &\leq \left\|\sqrt{T_n(\mu^n)}\nabla\omega^n\right\|_2 \left\|\sqrt{T_n(\mu^n)}\right\|_{2\frac{16-\lambda}{6+\lambda}} \\ &\leq \left\|\sqrt{T_n(\mu^n)}\nabla\omega^n\right\|_2 \|T_n(\mu^n)\|_{\frac{16-\lambda}{6+\lambda}}^2 . \end{aligned} \tag{170}$$

We see that $\frac{16-\lambda}{6+\lambda} \in (15/7, 8/3)$, and thus $\frac{16-\lambda}{6+\lambda} \leq \frac{8-\lambda^*}{3}$ for some $\lambda^* \in (0,1)$. From this and (168), (157) we obtain

$$\|T_n(\mu^n)\,\nabla\omega^n\|_{\frac{16-\lambda}{11}} \leq C(\lambda^{-1}). \tag{171}$$

Thus, having (171), (143), (73) and (77), we can conclude that

$$\int_0^T \|\partial_t\omega^n\|_{-1,\frac{16-\lambda}{11}}^{\frac{16-\lambda}{11}}\,dxdt \leq C(\lambda^{-1}). \tag{172}$$

## 9.2. PASSING TO THE LIMIT WITH N

Having (140), (141), (142), (143), (162), (167), (169), (172), we can find a subsequence (which we do not relabel) such that

$$v^n \rightharpoonup^* v \quad \text{weakly* in } L^\infty\left(0,T,L^2_{\mathrm{div}}\right) \cap L^2\left(0,T,W^{1,2}_{\mathrm{div}}(\Omega)\right), \tag{173}$$

$$\partial_t v^n \rightharpoonup \partial_t v \quad \text{weakly in } L^2\left(0,T,W^{-1,2}_{\mathrm{div}}(\Omega)\right), \tag{174}$$

$$b^n \rightharpoonup^* b \quad \text{weakly* in } L^q\left(0,T,W^{1,q}(\Omega)\right) \cap L^\infty(0,T,L^1(\Omega)) \text{ for all } q \in [1,5/4), \tag{175}$$

$$\partial_t b^n \rightharpoonup \partial_t b \quad \text{weakly in } \mathcal{M}\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,80/79), \tag{176}$$

$$\omega^n \rightharpoonup^* \omega \quad \text{weakly* in } L^2\left(0,T,W^{1,2}(\Omega)\right) \cap L^\infty(0,T,L^\infty(\Omega)), \tag{177}$$

$$\partial_t \omega^n \rightharpoonup \partial_t \omega \quad \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1,16/11). \tag{178}$$

Now, using the Aubin-Lions lemma 7, we conclude that for $\alpha \in (0,1)$ we have

$$v^n \to v \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega) \cap L^2_{\mathrm{div}}(\Omega)\right), \tag{179}$$

$$\omega^n \to \omega \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega)\right), \tag{180}$$

$$b^n \to b \qquad \text{strongly in } L^q\left(0,T,W^{\alpha,q}(\Omega)\right) \text{ for all } q \in \left[1,\frac{5}{4}\right). \tag{181}$$

We can extract subsequences that converge almost everywhere

$$v^n \to v \qquad \text{almost everywhere in } \Omega^T, \tag{182}$$

$$\omega^n \to \omega \qquad \text{almost everywhere in } \Omega^T, \tag{183}$$

$$b^n \to b \qquad \text{almost everywhere in } \Omega^T. \tag{184}$$

Moreover, from (160), (184), (143), (184) and Vitali lemma 4, we get

$$v^n \to v \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,10/3), \tag{185}$$

$$b^n \to b \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,5/3). \tag{186}$$

Now, we will identify the limit of (77) as $n \to \infty$. Notice that (171) implies

$$T_n(\mu^n)\nabla\omega^n \rightharpoonup \overline{\mu\nabla\omega} \quad \text{in } L^q(\Omega^T) \text{ for all } q \in [1,16/11). \tag{187}$$

Thus, using (179), (180), (178), we get

$$\langle \omega_{,t}, z\rangle - (\omega v, \nabla z) + \left(\overline{\mu\nabla\omega}, \nabla z\right) = -\kappa_2\left(\omega^2, z\right) \qquad \forall z \in W^{1,\infty}(\Omega) \text{ a.a. } t \in (0,T).$$

We need to show that $\overline{\mu\nabla\omega} = \mu\nabla\omega$ almost everywhere in $\Omega^T$. To do so we use Lemma 4 and (157), (183), (184) to get

$$T_n(\mu^n) \to \mu \text{ in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1,8/3). \tag{188}$$

Then, from (188), (177) and Lemma 5, we conclude

$$T_n(\mu^n)\nabla\omega^n \rightharpoonup \mu\nabla\omega \text{ in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in \left[1, \frac{8}{7}\right). \tag{189}$$

Using (189), (187) and the uniqueness of a weak limit, we get $\overline{\mu\nabla\omega} = \mu\nabla\omega$. Now, we will focus on obtaining a weak limit in (76). From (158), (183), (184) and Vitali lemma 4, for all $\alpha \in (0,1)$ we have

$$(T_n(\mu^n))^\alpha \to \mu^\alpha \text{ strongly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in \left[1, \frac{8}{3\alpha}\right). \tag{190}$$

From (164) and the fact that $\frac{81}{80} < \frac{80}{79}$, we get

$$(T_n(\mu^n))^\alpha \nabla b^n \rightharpoonup \overline{\mu^\alpha\nabla b} \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right) \text{ for all } \alpha \in [0,1]. \tag{191}$$

Our goal is to identify this limit for all $\alpha \in [0,1]$. We will proceed inductively. Define $h = \frac{1}{81}$, $\alpha_0 = 0$ and $\alpha_{i+1} = \alpha_i + h$. Notice that (191) holds for $\alpha_0$ due to (175). Assume that it holds for $\alpha_i$, that is

$$(T_n(\mu^n))^{\alpha_i} \nabla b^n \rightharpoonup \mu^{\alpha_i}\nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right). \tag{192}$$

Using (192), (190) with $\alpha = h$ and $q = \frac{4\cdot 81}{3}$ and lemma 5, we get

$$\begin{aligned}(T_n(\mu^n))^{\alpha_{i+1}} \nabla b^n &= (T_n(\mu^n))^h (T_n(\mu^n))^{\alpha_i} \nabla b^n \\ &\rightharpoonup \mu^h\mu^{\alpha_i}\nabla b = \mu^{\alpha_{i+1}}\nabla b \text{ weakly in } L^{\frac{324}{323}}\left(0,T;L^{\frac{324}{323}}(\Omega)\right).\end{aligned} \tag{193}$$

From (191), (193) and the uniqueness of the weak limit, we get

$$(T_n(\mu^n))^{\alpha_{i+1}} \nabla b^n \rightharpoonup \mu^{\alpha_{i+1}}\nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right).$$

Thus, setting $i = 81$, we get

$$T_n(\mu^n)\nabla b^n \rightharpoonup \mu\nabla b \text{ weakly in } L^{\frac{81}{80}}\left(0,T;L^{\frac{81}{80}}(\Omega)\right). \tag{194}$$

From (163), we deduce that $T_n(\mu^n)\nabla b^n \rightharpoonup \overline{\mu\nabla b}$ weakly in $L^q(0,T,L^q(\Omega))$ for all $q \in [1, 80/79)$. Thus, from the uniqueness of the weak limit and (194), we finally obtain

$$T_n(\mu^n)\nabla b \rightharpoonup \mu\nabla b \text{ weakly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1, 80/79). \tag{195}$$

From (183), (184), (29) and lemma 4, we can conclude that for all $\alpha \in (0,1)$

$$(T_k(\mu^n))^\alpha \rightharpoonup (T_k(\mu))^\alpha \text{ strongly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1,\infty). \tag{196}$$

From (140), (173), (196) with $\alpha = \frac{1}{2}$, Lemma 5 and the uniqueness of the weak limit, we have

$$\sqrt{T_k(\mu^n)}Dv^n \rightharpoonup \sqrt{T_k(\mu)}Dv \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{197}$$

Now, using (197), (196) with $\alpha = \frac{1}{2}$, (174), (143), (179), we can pass to the limit in (75) to get

$$\begin{aligned} \langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w\right) + \left(T_k\left(\mu\right) D(v), D(w)\right) = 0 \\ \forall w \in W^{1,2}_{\mathrm{div}}(\Omega) \text{ a.a. } t\in(0,T). \end{aligned} \tag{198}$$

The solution is defined on time interval $[0,T)$, however by repeating all previous steps it can also be attained on time interval $[0,T+\varepsilon)$. We will consider such extended solution to obtain stronger convergence results

$$\begin{aligned} \langle v_{,t}, w\rangle - \left(G_k\left(|v|^2\right) v\otimes v, \nabla w\right) + \left(T_k\left(\mu\right) D(v), D(w)\right) = 0 \\ \forall w \in W^{1,2}_{\mathrm{div}}(\Omega) \text{ a.a. } t\in(0,T+\varepsilon). \end{aligned} \tag{199}$$

First, notice that based on (179) and lemma 8 we have

$$v^n(t) \to v(t) \text{ in } L^2(\Omega) \text{ for almost all } t\in(0,T+\varepsilon). \tag{200}$$

Let us pick time $t^* \in (T, T+\varepsilon)$ such that (200) convergence holds. Now, let us set $w = v$ (which is a viable test function) in (199) and integrate from 0 to $t^*$

$$\|v(t^*)\|_2^2 + 2\int_{\Omega^{t^*}} T_k(\mu)|D(v)|^2 dxdt = \|v_0\|_2^2.$$

By setting $t = t^*$ in (139) (having in mind that it is valid for extended solution) and passing with $n\to\infty$, we get

$$\limsup_{n\to\infty}\left(\|v^n(t^*)\|_2^2 + 2\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n|^2 dxdt\right) = \|v_0\|_2^2.$$

By using (200), we get

$$\|v(t^*)\|_2^2 + 2\limsup_{n\to\infty}\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n|^2 dxdt = \|v_0\|_2^2. \tag{201}$$

By subtraction, we get

$$\limsup_{n\to\infty}\int_{\Omega^{t^*}} T_k(\mu^n)|Dv^n| dxdt = \int_{\Omega^{t^*}} T_k(\mu)|D(v)|^2 dxdt. \tag{202}$$

Thus, using (202) and (197) (again, having in mind that it can be attained up to time $T+\varepsilon$), we conclude that

$$\sqrt{T_k(\mu^n)}Dv^n \to \sqrt{T_k(\mu)}Dv \text{ strongly in } L^2\left(0,t^*;L^2(\Omega)\right)$$

and thus, due to the fact that $t^* \in (T, T+\varepsilon)$,

$$T_k(\mu^n)|Dv^n|^2 \to T_k(\mu)|Dv|^2 \text{ strongly in } L^1\left(0,T;L^1(\Omega)\right). \tag{203}$$

Having this, we can extract subsequence that converges almost everywhere

$$T_k(\mu^n)|Dv^n|^2 \to T_k(\mu)|Dv|^2 \text{ almost everywhere in } \Omega^T. \tag{204}$$

From (153), we have

$$\int_0^T \int_\Omega \frac{T_k(\mu^n)|Dv^n|^2}{T_k(\mu^n)+n^{-1}T_k(\mu^n)|Dv^n|^2}dxdt \leq C.$$

Using (204), (183), (184) and Fatou lemma, we get

$$\int_{\Omega^T} \frac{T_k(\mu)|Dv|^2}{T_k(\mu)}dxdt = \int_{\Omega^T} |Dv|^2 dxdt \leq C.$$

Now, we can strengthen (176). We see that, based on (76) and (203), (185), (186), (177), (195), we have

$$\begin{aligned}\int_0^T \langle \partial_t b^n, z\rangle dt = &\int_0^T (b^n v^n, \nabla z)\, dt - \int_0^T (T_n(\mu^n)\nabla b^n, \nabla z)\, dt - \int_0^T (b^n \omega^n, z)\, dt \\ &+ \int_0^T \left(\frac{T_k(\mu^n)\,|Dv^n|^2}{1+n^{-1}|Dv^n|^2}, z\right) dt \\ &\to \int_0^T (bv, \nabla z)\, dt - \int_0^T (\mu \nabla b, \nabla z)\, dt - \int_0^T (b\omega, z)\, dt \\ &+ \int_0^T \left(T_k(\mu)\,|Dv|^2, z\right) dt\end{aligned}$$

for all $z \in L^\infty\left(0,T,W^{1,q}(\Omega)\right)$, where $q \in (80,\infty]$. This means that

$$\partial_t b^n \rightharpoonup \partial_t b \text{ weakly in } L^1\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q \in [1, 80/79).$$

Using lemma 4 and (74), (73), (183), (184), we can deduce that

$$\begin{aligned}&\frac{1}{\sqrt{T_n(\mu^n)}\,(b^n)^{\frac{1+\lambda}{2}}} \to \frac{1}{\sqrt{\mu}\, b^{\frac{1+\lambda}{2}}} \\ &\qquad \text{strongly in } L^p(\Omega^T) \text{ for all } 1 \leq p < \infty \text{ and for all } 1 \leq \lambda < \infty.\end{aligned} \tag{205}$$

Combining (205), (195), we deduce that

$$\frac{\sqrt{T^n(\mu^n)}}{(b^n)^{\frac{1+\lambda}{2}}}\nabla b^n \rightharpoonup \frac{\sqrt{\mu}}{(b)^{\frac{1+\lambda}{2}}}\nabla b \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right).$$

Next, using (152), we can improve convergence result to

$$\frac{\sqrt{T^n(\mu^n)}}{(b^n)^{\frac{1+\lambda}{2}}}\nabla b^n \rightharpoonup \frac{\sqrt{\mu}}{(b)^{\frac{1+\lambda}{2}}}\nabla b \qquad \text{weakly in } L^2(\Omega^T).$$

Employing the same reasoning, one can show that

$$\frac{T^n(\mu^n)}{(b^n)^{\frac{1}{2}}}\nabla b^n \rightharpoonup \frac{\mu}{b^{\frac{1}{2}}}\nabla b \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right).$$

Finally, thanks to (160), (184) and Lemma 4, we get

$$\sqrt{b^n} \to \sqrt{b} \text{ strongly in } L^q(0,T,L^q(\Omega)) \text{ for all } q \in [1,10/3).$$

Thus, based on lemma 8, we see, that there exists a subsequence (which we do not relabel) such that

$$\sqrt{b^n(t)} \to \sqrt{b(t)} \text{ strongly in } L^q(\Omega) \text{ for all } q \in [1,10/3) \text{ and almost all times } t \in (0,T).$$

This gives us

$$\left(\sqrt{b^n(t)}, z\right) \to \left(\sqrt{b(t)}, z\right) \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ for almost all times } t \in (0,T).$$

Obtained convergence results are sufficient to pass to the limit with $n \to \infty$ in equations (75)-(77) and the inequality (80) to get Theorem 9.

# 10. PROOF OF MAIN THEOREM

For an arbitrary k, we denote by $(v^k, \omega^k, b^k)$ a solution to the problem (54)-(56), whose existence was established in Theorem 9.

## 10.1. K-INDEPENDENT ESTIMATES

First, from (54) tested with $v^k$ and (58), we get

$$\sup_{t\in(0,T)} \|v^k(t)\|_2^2 + \int_{\Omega^T} \left(1 + T_k(\mu^k)\right)|D(v^k)|^2 dxdt \le C. \tag{206}$$

Thus, using Korn inequality and interpolation inequality $\|f\|_{\frac{10}{3}}^{\frac{10}{3}} \le \|f\|_2^{\frac{4}{3}}\|f\|_{1,2}^2$, we get

$$\int_0^T \left\|v^k\right\|_{1,2}^2 + \left\|v^k\right\|_{\frac{10}{3}}^{\frac{10}{3}} dt \le C. \tag{207}$$

Moreover, from (58) we have

$$\int_0^T \left\|T_k(\mu^k)\right\|_{\frac{8-5\lambda}{3}}^{\frac{8-5\lambda}{3}} dt \le C(\lambda^{-1}). \tag{208}$$

Using (208), (206) and Hölder inequality, we get

$$\begin{aligned}
\left\|T_k(\mu^k)Dv^k\right\|_{\Omega^T,\frac{16-5\lambda}{11}} &\le \left\|\sqrt{T_k(\mu^k)}Dv^k\right\|_{\Omega^T,2}\left\|\sqrt{T_k(\mu^k)}\right\|_{\Omega^T,2\frac{16-5\lambda}{6+5\lambda}} \\
&\le \left\|\sqrt{T_k(\mu^k)}Dv^k\right\|_{\Omega^T,2}\left\|T_k(\mu^k)\right\|^2_{\Omega^T,\frac{16-5\lambda}{6+5\lambda}} \le C(\lambda^{-1}),
\end{aligned} \tag{209}$$

due to $\frac{16-5\lambda}{6+5\lambda} \in (1, 8/3)$. Notice that from (58) and (52), it follows that

$$\int_{\Omega^T} |b^k|^{\frac{8-5\lambda}{3}} \le C(\lambda^{-1}). \tag{210}$$

Using (58), (210), (52) and the Hölder inequality, we get

$$\begin{aligned}
\|\nabla b\|_{2-\lambda} &\le \left\|\sqrt{\frac{\mu^k}{\left(b^k\right)^{1+\lambda}}}\nabla b^k\right\|_2 \left\|\sqrt{\frac{\left(b^k\right)^{1+\lambda}}{\mu^k}}\right\|_{\frac{2(2-\lambda)}{\lambda}} \\
&\le C\left\|\sqrt{\frac{\mu^k}{\left(b^k\right)^{1+\lambda}}}\nabla b^k\right\|_2 \left\|b^k\right\|^{\frac{\lambda}{2}}_{2-\lambda} \le C(\lambda^{-1}).
\end{aligned} \tag{211}$$

Notice that for any $\lambda \in (0,1)$ we can find $\lambda_1, \lambda_2 \in (0,1)$ such that $\frac{7}{8-\lambda} = \frac{1}{2-\lambda_1} + \frac{3}{8-5\lambda_2}$. Additionally, $\lambda_1, \lambda_2 \to 0^+$ as $\lambda \to 0^+$. Thus, by the Hölder inequality and (52), (211), (210), we get

$$\left\|\mu^k \nabla b^k\right\|_{\Omega^T,\frac{8-\lambda}{7}} \le \left\|\nabla b^k\right\|_{\Omega^T,2-\lambda_1}\left\|\mu^k\right\|_{\Omega^T,\frac{8-5\lambda_2}{3}} \le C\left(\lambda_1^{-1},\lambda_2^{-1}\right) \le C(\lambda^{-1}). \tag{212}$$

From (58), (211), (210) and (212) we have

$$\begin{aligned}
\sup_{t\in(0,T)}\left(\|b^k(t)\|_1 + \|\ln b^k(t)\|_1\right) + \int_0^T \left(\left\|\nabla b^k\right\|^{2-\lambda}_{2-\lambda} + \left\|b^k\right\|^{\frac{8-5\lambda}{3}}_{\frac{8-5\lambda}{3}}\right) dt \\
+ \int_0^T \left\|\mu^k \nabla b^k\right\|^{\frac{8-\lambda}{7}}_{\frac{8-\lambda}{7}} dt \le C(\lambda^{-1}).
\end{aligned} \tag{213}$$

Based on equation (55), we have

$$\begin{aligned}
\left\|\partial_t b^k\right\|_{-1,\frac{8-\lambda}{7}} &= \sup_{\varphi\in W^{1,\left(\frac{8-\lambda}{7}\right)'}(\Omega):\|\varphi\|_{W^{1,\left(\frac{8-\lambda}{7}\right)'}(\Omega)}=1} \left|\left\langle \partial_t b^k, \varphi\right\rangle\right| \\
&\le \sup_{\substack{\varphi\in W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega)\\ \|\varphi\|_{1,\frac{8-\lambda}{1-\lambda}}=1}} \left[\left|\left(b^k v^k, \nabla\varphi\right)\right| + \left|\left(\mu^k b^k, \nabla\varphi\right)\right| + \left|\left(\omega^k b^k, \varphi\right)\right| + \left|\left(T_k(\mu^k)|Dv^k|^2, \varphi\right)\right|\right]
\end{aligned}$$

Let us note that $W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega) \subset W^{1,8}(\Omega) \subset C^{0,\frac{5}{8}}(\Omega)$, and thus, by Hölder and Young inequalities, we have

$$\begin{aligned}
\left\|\partial_t b^k\right\|_{-1,\frac{8-\lambda}{7}} \leq & \sup_{\substack{\varphi \in W^{1,\frac{8-\lambda}{1-\lambda}}(\Omega) \\ \|\varphi\|_{1,\frac{8-\lambda}{1-\lambda}}=1}} \Bigg[ \left\|v^k\right\|_{\frac{10}{3}} \left\|b^k\right\|_{\frac{40}{23}} \|\nabla\varphi\|_8 + \left\|\mu^k \nabla b^k\right\|_{\frac{8-\lambda}{7}} \|\nabla\varphi\|_{\frac{8-\lambda}{1-\lambda}} \\
& \qquad + \left\|\omega^k\right\|_\infty \left\|b^k\right\|_1 \|\varphi\|_\infty + \left\|T_k(\mu^k)|Dv^k|^2\right\|_1 \|\varphi\|_\infty \Bigg] \\
& \leq C\left( \left\|v^k\right\|_{\frac{10}{3}}^{\frac{10}{3}} + \left\|b^k\right\|_{\frac{40}{23}}^{\frac{40}{23}} + \left\|\mu^k \nabla b^k\right\|_{\frac{8-\lambda}{7}}^{\frac{8-\lambda}{7}} + \left\|b^k\right\|_1 + \left\|T_k(\mu^k)|Dv^k|^2\right\|_1 + 1 \right).
\end{aligned}$$

Finally, by (213), (210), (207), (206), we get

$$\int_0^T \left\|\partial_t b^k\right\|_{W^{-1,\frac{8-\lambda}{7}}(\Omega)} dt \leq C(\lambda^{-1}). \tag{214}$$

Next, notice, that (56), tested with $\omega^k$, and (52) imply

$$\sup_{t\in(0,T)} \left\|\omega^k(t)\right\|_\infty + \int_{\Omega^T} b^k |\nabla\omega^k|^2 dxdt \leq C. \tag{215}$$

Using the Hölder inequality and (215), (210), we get

$$\begin{aligned}
\left\|b^k \nabla\omega^k\right\|_{\frac{16-5\lambda}{11}} &\leq \left\|\sqrt{b^k}\nabla\omega^k\right\|_2 \left\|\sqrt{b^k}\right\|_{2\frac{16-5\lambda}{6+5\lambda}} \\
&\leq \left\|\sqrt{b^k}\nabla\omega^k\right\|_2 \left\|b^k\right\|_{\frac{16-5\lambda}{6+5\lambda}}^2 \leq C(\lambda^{-1}).
\end{aligned} \tag{216}$$

Using (213), (52) and (216), we get

$$\left\|\nabla(b^k\omega^k)\right\|_{\frac{16-5\lambda}{11}} \leq C(\lambda^{-1}) \tag{217}$$

and

$$\left\|\nabla\left(\frac{b^k\omega^k}{b^k+1}\right)\right\|_{\frac{16-5\lambda}{11}} \leq C(\lambda^{-1}). \tag{218}$$

Also, with the help of (218), (210), (52), we can write

$$\int_0^T \left\|b^k\omega^k\right\|_{W^{1,\frac{16-5\lambda}{11}}(\Omega)}^{\frac{16-5\lambda}{11}} dt \leq C(\lambda^{-1}). \tag{219}$$

Using the equation (56) and inequalities (52), (207), (216), we deduce (in a similar way as in (214)) the following:

$$\int_0^T \left\|\partial_t\omega^k\right\|_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)}^{\frac{16-5\lambda}{11}} dt \leq C(\lambda^{-1}). \tag{220}$$

## 10.2. RECONSTRUCTION OF PRESSURE

We will show that there exists pressure $p_k \in L^2(0,T,L^2(\Omega))$ such that

$$\langle v^k_{,t}, w\rangle - \left(G_k\left(\left|v^k\right|^2\right)v^k \otimes v^k, \nabla w\right) + \left(T_k\left(\mu^k\right)D(v^k), D(w)\right) = \left(p^k, \operatorname{div} w\right) \qquad \forall w \in W^{1,2}(\Omega) \text{ and almost all } t \in (0,T). \tag{221}$$

Combining Lemma 2 with (47) and (29), (31) we get

$$p^k_1 = \mathcal{L}(T_k(\mu^k)Dv^k) \in L^2(\Omega) \qquad \text{for almost all } t \in (0,T), \tag{222}$$
$$p^k_2 = \mathcal{L}(-G_k(|v^k|^2)v^k \otimes v^k) \in L^2(\Omega^T) \qquad \text{for almost all } t \in (0,T), \tag{223}$$

which are uniquely defined for a fixed $k$. Additionally, using the estimates (207), (209), we have

$$\left\|p^k_1\right\|_{\frac{16-5\lambda}{11}} \le C\left\|T_k(\mu^k)Dv^k\right\|_{\frac{16-5\lambda}{11}} \qquad \text{for almost all } t \in (0,T), \tag{224}$$
$$\left\|p^k_2\right\|_{\frac{5}{3}} \le C\left\|v^k\right\|^2_{\frac{10}{3}} \qquad \text{for almost all } t \in (0,T), \tag{225}$$

Moreover, the following equalities hold:

$$\left(p^k_1, \Delta\phi\right) = \left(T_k(\mu^k)D(v^k), \nabla(\nabla\phi)\right) \qquad \text{for all } \phi \in W^{2,2}(\Omega), \tag{226}$$
$$\left(p^k_2, \Delta\phi\right) = -\left(G_k\left(\left|v^k\right|^2\right)v^k \otimes v^k, \nabla^2\phi\right) \qquad \text{for all } \phi \in W^{2,2}(\Omega), \tag{227}$$
$$\int_\Omega p^k_1 dx = \int_\Omega p^k_2 dx = 0. \tag{228}$$

Let $w \in W^{1,2}(\Omega)$. It can be decomposed (using Helmholtz decomposition) in the following way:

$$w = \nabla\varphi + \nabla \times A,$$

where $\varphi, A \in W^{2,2}(\Omega)$. Since $\operatorname{div}(\nabla \times A) = 0$, from (54) we have

$$\langle v^k_{,t}, \nabla \times A\rangle - \left(G_k\left(\left|v^k\right|^2\right)v^k \otimes v^k, \nabla(\nabla \times A)\right) + \left(T_k\left(\mu^k\right)D(v^k), D(\nabla \times A)\right) = 0. \tag{229}$$

We also see that due to $\operatorname{div} v^k = 0$, we have

$$\langle v^k_{,t}, \nabla\varphi\rangle = 0. \tag{230}$$

Since $\operatorname{div}(\nabla \times A) = 0$, we can write

$$\left(p^k_1, \operatorname{div}(\nabla \times A)\right) = 0, \qquad \left(p^k_2, \operatorname{div}(\nabla \times A)\right) = 0. \tag{231}$$

Thus, summing (231), (230), (229), (227), (226), and using the fact that

$$\left(T_k(\mu^k)D(v^k),\nabla(\nabla\phi)\right)=\left(T_k(\mu^k)D(v^k),D(\nabla\phi)\right),$$

we get

$$\begin{aligned}\langle v^k_{,t},\nabla\phi+\nabla\times A\rangle-\left(G_k\left(\left|v^k\right|^2\right)v^k\otimes v^k,\nabla\left(\nabla\phi+\nabla\times A\right)\right)\\+\left(T_k\left(\mu^k\right)D(v^k),D\left(\nabla\phi+\nabla\times A\right)\right)=\left(p^k,\operatorname{div}\left(\nabla\phi+\nabla\times A\right)\right),\end{aligned}$$

where $p^k=p_1^k+p_2^k$. The obtained equality is exactly (221).
Next, using (224), (225) and estimates (207), (209), we deduce

$$\int_0^T\left(\left\|p_1^k\right\|_{\frac{16-\lambda}{11}}^{\frac{16-5\lambda}{11}}+\left\|p_2^k\right\|_{\frac{5}{3}}^{\frac{5}{3}}\right)dt\le C(\lambda^{-1}). \tag{232}$$

Now, based on the equation (221) for $\lambda\in(0,1)$ and proceeding as in (214), we have

$$\left\|\partial v_t^k\right\|_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)}\le C\left(\left\|v^k\right\|_{\frac{10}{3}}^2+\left\|T_k(\mu^k)Dv^k\right\|_{\frac{16-5\lambda}{11}}+\|p_1\|_{\frac{16-5\lambda}{11}}+\|p_2\|_{\frac{5}{3}}\right). \tag{233}$$

Consequently by estimates (206), (207), (232), we have

$$\int_0^T\left\|\partial_t v^k\right\|_{W^{-1,\frac{16-5\lambda}{11}}(\Omega)}^{\frac{16-5\lambda}{11}}dt\le C(\lambda^{-1}). \tag{234}$$

## 10.3. TAKING THE LIMIT $K\to\infty$

Based on (206), (207), (234), (213), (214), (215), (220), (210), (232), we can deduce the existence of a subsequence (which we do not relabel) such that

$$v^k\rightharpoonup^* v \quad \text{weakly* in } L^\infty\left(0,T,L^2_{\mathrm{div}}(\Omega)\right)\cap L^2\left(0,T,W^{1,2}_{\mathrm{div}}(\Omega)\right), \tag{235}$$

$$v^k\rightharpoonup v \quad \text{weakly in } L^{\frac{10}{3}}\left(0,T,L^{\frac{10}{3}}(\Omega)\right), \tag{236}$$

$$\partial_t v^k\rightharpoonup\partial_t v \quad \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q\in\left[1,\frac{16}{11}\right), \tag{237}$$

$$b^k\rightharpoonup^* b \quad \text{weakly* in } L^q\left(0,T,W^{1,q}(\Omega)\right)\cap L^\infty\left(0,T,L^1(\Omega)\right) \text{ for all } q\in[1,2), \tag{238}$$

$$\partial_t b^k\rightharpoonup\partial_t b \quad \text{weakly in } \mathcal{M}\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q\in[1,8/7), \tag{239}$$

$$\omega^k\rightharpoonup^*\omega \quad \text{weakly* in } L^\infty\left(0,T,L^\infty(\Omega)\right), \tag{240}$$

$$\partial_t\omega^k\rightharpoonup\partial_t\omega \quad \text{weakly in } L^q\left(0,T,W^{-1,q}(\Omega)\right) \text{ for all } q\in[1,16/11), \tag{241}$$

$$b^k\rightharpoonup b \quad \text{weakly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q\in[1,8/3), \tag{242}$$

$$p_1^k\rightharpoonup p_1 \quad \text{weakly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q\in[1,16/11), \tag{243}$$

$$p_2^k\rightharpoonup p_2 \quad \text{weakly in } L^{\frac{5}{3}}\left(0,T,L^{\frac{5}{3}}(\Omega)\right). \tag{244}$$

From Aubin-Lions lemma, we conclude that for $\alpha \in (0,1)$

$$v^n \to v \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega)\cap L^2_{\mathrm{div}}(\Omega)\right), \tag{245}$$
$$b^n \to b \qquad \text{strongly in } L^2\left(0,T,W^{\alpha,2}(\Omega)\right). \tag{246}$$

We can extract subsequences that converge almost everywhere

$$v^k \to v \qquad \text{almost everywhere in } \Omega^T, \tag{247}$$
$$b^k \to b \qquad \text{almost everywhere in } \Omega^T. \tag{248}$$

Thus, based on inequalities (213), (207) and Vitali lemma 4, we have

$$v^k \to v \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,10/3), \tag{249}$$
$$b^k \to b \qquad \text{strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q \in [1,8/3). \tag{250}$$

Using (219), (218), (250), (240), lemma 5 and the uniqueness of the weak limit, we get

$$b^k\omega^k \rightharpoonup b\omega \qquad \text{weakly in } L^q\left(0,T,W^{1,q}(\Omega)\right) \text{ for all } q \in [1,16/11), \tag{251}$$
$$\frac{b^k\omega^k}{1+b^k} \rightharpoonup \frac{b\omega}{1+b} \qquad \text{weakly in } L^q\left(0,T,W^{1,q}(\Omega)\right) \text{ for all } q \in [1,16/11). \tag{252}$$

Our goal is to strengthen convergence result for $\omega^k$. To achieve this, we employ Div-Curl lemma (see lemma 3). Let us define two 4-vectors

$$a^k := \left(\omega^k, \omega^k v^k - \mu^k \nabla\omega^k\right), \qquad c^k := \left(b^k\left(1+b^k\right)^{-1}\omega^k, 0,0,0\right).$$

Using (52), (207) and (216), we get

$$\left\|a^k\right\|_{L^{\frac{16-5\lambda}{11}}(\Omega^T)} + \left\|c^k\right\|_{L^\infty(\Omega^T)} \le C(\lambda^{-1}).$$

From the equation (56), the maximum principle (52) and (218), we have

$$\left\|\mathrm{div}_{t,x}\, a^k\right\|_{L^\infty(\Omega^T)} = \left\|\partial_t\omega^k + \mathrm{div}\left(\omega^k v^k\right) - \mathrm{div}\left(\mu^k\nabla\omega^k\right)\right\|_{L^\infty(\Omega^T)} = \kappa_2\left\|\left(\omega^k\right)^2\right\|_{L^\infty(\Omega^T)} \le C$$

and

$$\left\|\nabla_{t,x}c^k - \left(\nabla_{t,x}c^k\right)^T\right\|_{L^1(\Omega^T)} \le C\left\|\nabla\left(\frac{b^k\omega^k}{1+b^k}\right)\right\|_{L^1(\Omega^T)} \le C.$$

Using (240), (249), (216), (52) in case of convergence of $a^k$ and (252), (52) combined with the uniqueness of the weak limit in case of convergence of $c^k$, we get

$$a^k \rightharpoonup a = \left(\omega, \omega v - \overline{\mu\nabla\omega}\right) \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in [1,16/11),$$
$$c^k \rightharpoonup^* c = \left(b\left(1+b\right)^{-1}\omega, 0,0,0\right) \qquad \text{weakly* in } L^\infty(\Omega^T).$$

Thus, using Div-Curl lemma 3, we get

$$\frac{b^k|\omega^k|^2}{1+b^k} \rightharpoonup \frac{b|\omega|^2}{1+b} \text{ in the sense of distributions.} \tag{253}$$

However, we see that the sequence $\frac{b^k|\omega^k|^2}{1+b^k}$ is bounded in $L^\infty(\Omega^T)$, so a weak sequence can be extracted. Using the uniqueness of the weak limit, we get

$$\frac{b^k|\omega^k|^2}{1+b^k} \rightharpoonup^* \frac{b|\omega|^2}{1+b} \text{ weakly* in } L^\infty(\Omega^T). \tag{254}$$

Using (254) and (250), we can deduce that

$$\begin{aligned} \int_{\Omega^T} \left(b^k \omega^k\right)^2 dx &= \int_{\Omega^T} b^k \left(b^k+1\right) \frac{b^k|\omega^k|^2}{1+b^k} dx \\ &\to \int_{\Omega^T} b\,(b+1)\, \frac{b|\omega|^2}{1+b} dx = \int_{\Omega^T} (b\omega)^2\, dx. \end{aligned} \tag{255}$$

From (250) and (240), we get

$$b^k\omega^k \rightharpoonup b\omega \text{ weakly in } L^2(\Omega^T). \tag{256}$$

And using (255) and (256), we get

$$b^k\omega^k \to b\omega \text{ strongly in } L^2(\Omega^T). \tag{257}$$

Consequently, for a subsequence we have

$$b^k\omega^k \to b\omega \text{ almost everywhere in } \Omega^T. \tag{258}$$

Using Vitali lemma 4, (258), (248) and (52), we get

$$\omega^k = \frac{b^k\omega^k}{b^k} \to \frac{b\omega}{b} = \omega \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,\infty). \tag{259}$$

Having the above convergence and (52), it is easy to see that

$$\frac{1}{\omega^k} \to \frac{1}{\omega} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,\infty). \tag{260}$$

Using (260) and (250), we conclude that

$$\mu^k \to \mu = \frac{b}{\omega} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1,8/3). \tag{261}$$

Also, there exists a subsequence (which we do not relabel) such that

$$\mu^k \to \mu \text{ almost everywhere in } \Omega^T. \tag{262}$$

From (261) combined with (238), we deduce

$$\mu^k \nabla b^k \rightharpoonup \mu \nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 8/7). \tag{263}$$

Thanks to (206), we can deduce that $\sqrt{T_k(\mu^k)}D(v^k) \rightharpoonup \overline{\sqrt{\mu}D(v)}$ in $L^2(\Omega^T)$, and thus, by (235), (261) and the uniqueness of the weak limit, we have

$$\sqrt{T_k(\mu^k)}D(v^k) \rightharpoonup \sqrt{\mu}D(v) \qquad \text{weakly in } L^2(\Omega^T). \tag{264}$$

Again, using (262), (213), (52) and lemma 4, we conclude that

$$\sqrt{T_k\left(\mu^k\right)} \to \sqrt{\mu} \text{ strongly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/3). \tag{265}$$

Now, from (265), (264) and weak-strong convergence lemma 5, we can deduce

$$T_k\left(\mu^k\right) Dv^k \rightharpoonup \mu Dv \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11). \tag{266}$$

Using (260), (251) and (238), we get

$$\mu^k \nabla b^k = \frac{\nabla\left(b^k \omega^k\right)}{\omega^k} - \nabla b^k \rightharpoonup \frac{\nabla(b\omega)}{\omega} - \nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q \in [1, 16/11).$$

The convergence results obtained above are sufficient to pass to the limit in (54)-(56) to get (24), (26), (28). Now, we will focus on obtaining (25). Let us denote by $E^k := |v^k|^2/2 + b^k$. Let us set $w = v^k z$, $z \in W^{1,\infty}(\Omega)$ in (221) and sum it with (55) to get

$$\begin{aligned}\left\langle E^k_t, z\right\rangle - \left(\left(E^k + p^k\right) v^k, \nabla z\right) + \left(\mu^k \nabla b^k, \nabla z\right) + \left(T_K\left(\mu^k\right) D\left(v^k\right) v^k, \nabla z\right) \\ = \left(-b^k \omega^k, z\right) + \frac{1}{2}\left(\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k, \nabla z\right).\end{aligned} \tag{267}$$

First, let us observe that by (207), (31), (32), the sequence $\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k$ is bounded in $L^{\frac{10}{9}}(\Omega^T)$, and thus there exists a weakly convergent subsequence (which we do not relabel):

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \rightharpoonup \overline{0} \text{ weakly in } L^{\frac{10}{9}}(\Omega^T)\,. \tag{268}$$

Using (247), (31), (32), we obtain

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \to 0 \text{ almost everywhere in } \Omega^T\,. \tag{269}$$

Thus, by (269), (268) and the Egorov theorem, we conclude that

$$\left(2G_k\left(|v^k|^2\right)|v^k|^2 - |v^k|^2 - \Gamma_k\left(|v^k|^2\right)\right) v^k \rightharpoonup 0 \text{ weakly in } L^{\frac{10}{9}}(\Omega^T)\,. \tag{270}$$

From (247) and (248), we have

$$E^k \to E \quad \text{almost everywhere in } \Omega^T. \tag{271}$$

From (207), (213) and (271) combined with the Egorov theorem, we have

$$E^k \rightharpoonup \overline{E} = E \qquad \text{weakly in } L^{\frac{5}{3}}(\Omega^T). \tag{272}$$

Now, we see that due to (207) and (272), $v^k E^k$ is bounded in $L^{\frac{10}{9}}(\Omega^T)$, and thus has a weakly convergent subsequence. This, Egorov lemma, (271) and (247) imply that

$$v^k E^k \rightharpoonup vE \qquad \text{weakly in } L^{\frac{10}{9}}(\Omega^T)\,. \tag{273}$$

Finally, using (266), (249), (243), (244) and weak-strong convergence lemma 5, we get

$$T_k(\mu^k)D(v^k)v^k \rightharpoonup \mu D(v)v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right), \tag{274}$$

$$p_1^k v^k \rightharpoonup p_1 v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{80}{79}\right), \tag{275}$$

$$p_2^k v^k \rightharpoonup p_2 v \qquad \text{weakly in } L^q(\Omega^T) \text{ for all } q \in \left[1, \frac{10}{9}\right). \tag{276}$$

From the equation (267) and (270), (273)-(276), (263), (257), recalling that weakly convergent sequence is bounded, we deduce that

$$\int_0^T \left\|\partial_t E^k\right\|^q_{W^{-1,q}(\Omega)} dt \le C \text{ for all } q \in [1, 80/79).$$

Thus, one can pass to the limit in (267) to get (25).

## 10.4. ATTAINMENT OF INITIAL DATA

In this part, we focus on obtaining initial conditions in a similar fashion as presented in [4]. We start with $v$. Let us test equation (54) with $\varphi \in D(\Omega)$ such that $\operatorname{div}\varphi = 0$ and integrate from 0 to $t$

$$\left(v^k(t), \varphi\right) - (v_0, \varphi) - \int_0^t \left(v^k \otimes v^k, D\varphi\right) dt + \int_0^t \left(T_k(\mu^k)Dv^k, D\varphi\right) dx = 0. \tag{277}$$

Using (249) and lemma 8, we obtain

$$v^k(t) \to v(t) \text{ in } L^2(\Omega) \text{ for almost all } t \in (0, T). \tag{278}$$

Using (266), (249) and (278), we can pass to the limit in (277)

$$(v(t), \varphi) - (v_0, \varphi) - \int_0^t (v \otimes v, D\varphi)\, dt + \int_0^t (\mu Dv, D\varphi)\, dt = 0 \text{ for almost all } t \in (0, T).$$

From this, we deduce

$$\lim_{t\to 0^+}(v(t),\varphi) = (v_0,\varphi)\,. \tag{279}$$

The equality also holds for $\varphi \in L^2_{\mathrm{div}}(\Omega)$. Indeed, let $\{\varphi_j\}$ be a sequence of smooth functions such that $\varphi_j \to \varphi$ in $L^2(\Omega)$. First, let us observe that by (235) we have

$$\lim_{j\to\infty}\lim_{t\to 0^+}|\left(v(t),\varphi_j-\varphi\right)| \le \sup_{t\in(0,T)}\|v(t)\|_2 \lim_{j\to\infty}\|\varphi_j-\varphi\|_2 = 0. \tag{280}$$

Now, using (279) we have

$$\lim_{j\to\infty}\lim_{t\to 0^+}\left(v(t),\varphi_j\right) = \lim_{j\to\infty}\lim_{t\to 0^+}(v(t),\varphi) + \lim_{j\to\infty}\lim_{t\to 0^+}\left(v(t),\varphi_j-\varphi\right) = (v_0,\varphi)\,.$$

From this and (280) we deduce that (279) also holds for $\varphi$ in $L^2_{\mathrm{div}}(\Omega)$.
Now, testing equation (54) with $v^k$ and integrating from 0 to $t$, we get

$$\|v^k(t)\|_2^2 + 2\int_0^t \left(T_k(\mu^k)Dv^k, Dv^k\right)dt = \|v_0\|_2^2.$$

Next, omitting second term of the left hand side and passing to the limit with $k\to\infty$ with the use of (278), we get

$$\|v(t)\|_2^2 \le \|v_0\|_2^2 \text{ for a.a. } t\in(0,T). \tag{281}$$

Using (279) and (281), we conclude that

$$\begin{aligned}\lim_{t\to 0^+}\|v(t)-v_0\|_2^2 &= \lim_{t\to 0^+}\left(\|v(t)\|_2^2+\|v_0\|_2^2-2\,(v(t),v_0)\right)\\ &\le \|v_0\|_2^2+\|v_0\|_2^2-2\,(v_0,v_0)=0.\end{aligned} \tag{282}$$

Similarly, we can show attainment of initial data for $\omega$.
Now, we will concentrate on showing attainment of initial data by $b$. Before we proceed further, we will establish more convergence results. By (262), (213), (52) and lemma 4, we have

$$\sqrt{\mu^k} \to \sqrt{\mu} = \sqrt{\frac{b}{\omega}} \text{ strongly in } L^q(\Omega^T) \text{ for all } q\in[1,16/3). \tag{283}$$

From (283) combined with (238), we deduce

$$\sqrt{\mu^k}\nabla b^k \rightharpoonup \sqrt{\mu}\nabla b \text{ weakly in } L^q(\Omega^T) \text{ for all } q\in[1,16/11). \tag{284}$$

From (238), (248), Lemma 4, we conclude that

$$\sqrt{b^k} \to \sqrt{b} \text{ strongly in } L^q\left(0,T,L^q(\Omega)\right) \text{ for all } q\in[1,4), \tag{285}$$

Using (285) and lemma 8, we get

$$\sqrt{b^k(t)} \to \sqrt{b(t)} \text{ in } L^2(\Omega) \text{ for almost all } t\in(0,T). \tag{286}$$

Now, using (59) for almost all times $t \in (0,T)$, we have

$$\left(\sqrt{b^k(t)}, \varphi\right) - \int_0^t \left(\sqrt{b^k} v^k, \nabla\varphi\right) d\tau + \int_0^t \left(\frac{\sqrt{\omega^k}}{2}\sqrt{\mu^k}\nabla b^k, \nabla\varphi\right) d\tau$$
$$\geq \frac{1}{2}\int_0^t \left(\sqrt{b^k}\omega^k, \varphi\right) d\tau + \left(\sqrt{b_0^k}, \varphi\right) \quad \forall\varphi \in D(\Omega), \varphi \geq 0.$$

Using (285), (284), (249), (238), (261), (286) and letting $k \to \infty$, we get

$$\left(\sqrt{b(t)}, \varphi\right) - \int_0^t \left(\sqrt{b}v, \nabla\varphi\right) d\tau + \int_0^t \left(\frac{\sqrt{\omega}}{2}\sqrt{\mu}\nabla b, \nabla\varphi\right) d\tau$$
$$\geq \frac{1}{2}\int_0^t \left(\sqrt{b}\omega, \varphi\right) d\tau + \left(\sqrt{b_0}, \varphi\right) \quad \forall\varphi \in D(\Omega), \varphi \geq 0 \text{ for almost all } t \in (0,T).$$

Finally, letting $t \to 0^+$, we get

$$\liminf_{t\to 0^+} \left(\sqrt{b(t)}, \varphi\right) \geq \left(\sqrt{b_0}, \varphi\right) \quad \forall\varphi \in D(\Omega), \varphi \geq 0. \tag{287}$$

Note that the obtained inequality is also valid for $\varphi \in L^2(\Omega)$, as before in (279), due to density argument. Now, setting $z = I_{\{0\leq\tau\leq t\}}$ in (267) and integrating from 0 to $t$, we get

$$\int_0^t \langle \partial_t E^k, 1\rangle d\tau = -\int_0^t \left(b^k\omega^k, 1\right) d\tau.$$

Thus,

$$\int_\Omega b^k(x,t)dx + \int_\Omega |v^k(x,t)|^2 dx = -\int_0^t \left(b^k\omega^k, 1\right) d\tau + \int_\Omega b_0^k(x)dx + \int_\Omega |v_0(x)|^2 dx.$$

Using (286), (278), (257) and letting letting $k \to \infty$, we obtain

$$\int_\Omega b(x,t)dx + \int_\Omega |v(x,t)|^2 dx = -\int_0^t (b\omega, 1)\, d\tau + \int_\Omega b_0(x)dx + \int_\Omega |v_0(x)|^2 dx$$

for almost all $t \in (0,T)$. Finally, letting $t \to 0^+$, we get

$$\limsup_{t\to 0^+} \left(\int_\Omega b(x,t)dx + \int_\Omega |v(x,t)|^2 dx\right) = \int_\Omega b_0(x)dx + \int_\Omega |v_0(x)|^2 dx.$$

Thus, employing (282), we get

$$\limsup_{t\to 0^+} \int_\Omega b(x,t)dx = \int_\Omega b_0(x)dx. \tag{288}$$

Notice that by (288) and (287) we get

$$\begin{aligned}
\limsup_{t\to 0^+} \|\sqrt{b(t)} - \sqrt{b_0}\|_2^2 &= \limsup_{t\to 0^+} \left(\|b(t)\|_1 + \|b_0\|_1 - 2\left(\sqrt{b(t)}, \sqrt{b_0}\right)\right) \\
&\leq \|b_0\|_1 + \|b_0\|_1 + 2\limsup_{t\to 0^+} \left(-\left(\sqrt{b(t)}, \sqrt{b_0}\right)\right) \\
&\leq 2\|b_0\|_1 - 2\liminf_{t\to 0^+} \left(\sqrt{b(t)}, \sqrt{b_0}\right) \\
&\leq 2\|b_0\|_1 - 2\left(\sqrt{b_0}, \sqrt{b_0}\right) \leq 0.
\end{aligned}$$

Now, with the help of (238) it is straightforward to show attainment of initial data

$$\begin{aligned}\lim_{t\to 0^+}\|b(t)-b_0\|_1 &\le \lim_{t\to 0^+}\|\sqrt{b(t)}-\sqrt{b_0}\|_2\|\sqrt{b(t)}+\sqrt{b_0}\|_2\\ &\le 2\sup_{\tau\in(0,T)}\|b(\tau)\|_1^{1/2}\lim_{t\to 0^+}\|\sqrt{b(t)}-\sqrt{b_0}\|_2\\ &= 0.\end{aligned}$$

This concludes the proof of the main theorem.

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# ISOMETRIES IN HAUSDORFF METRIC

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**Abstract:** In this paper, we characterize the isometries of nonempty compact subsets of certain uniquely geodesic metric spaces, endowed with the Hausdorff metric.
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## 1. INTRODUCTION

Let $(X,d)$ be a metric space. On the family $\mathfrak{B}(X)$ of nonempty, closed, bounded subsets of $X$, we define the map $d_{\mathrm{H}} \colon \mathfrak{B}(X) \times \mathfrak{B}(X) \to [0,\infty]$ by the formula:

$$\forall A,B \in \mathfrak{B}(X) \quad d_{\mathrm{H}}(A,B) := \inf\left\{r \geq 0 \colon A \subseteq H(B,r) \quad \text{and} \quad B \subseteq H(A,r)\right\},$$

the so-called *Hausdorff metric* on $\mathfrak{B}(X)$, where $H(A,r)$ denotes the $r-$hull of $A$. We shall restrict this metric to the family $\mathfrak{C}(X)$ of all nonempty compact subsets of $X$.

It is well-known that given an isometry $i$ of $(X,d)$, the function $I$ defined by the formula:

$$\forall K \in \mathfrak{C}(X) \quad I(K) := i[K] = \left\{i(k) \colon k \in K\right\}$$

is an isometry of the space $(\mathfrak{C}(X), d_{\mathrm{H}})$. The following question is much more interesting: what does one need to assume about the space $(X,d)$, so that all isometries of $(\mathfrak{C}(X), d_{\mathrm{H}})$ can be obtained by the above formula from isometries of $(X,d)$.

The main purpose of this paper is to answer this question for certain uniquely geodesic metric spaces by proving the following theorem:

**Theorem 1.** *Let $(X,d)$ be a uniquely geodesic, geodesically complete metric space, in which geodesics do not split. Let $I$ be an isometry of $(\mathfrak{C}(X), d_{\mathrm{H}})$. Then there exists an isometry $i$ of $(X,d)$ such that:*

$$\forall K \in \mathfrak{C}(X) \quad I(K) = i[K].$$

In the late 70's and early 80's, several authors started to investigate relations between the isometries of the Euclidean space $\mathbb{E}^n$ and those of the space $(\mathcal{C}(\mathbb{E}^n), d_{\mathrm{H}})$, where $\mathcal{C}(\mathbb{E}^n)$ denotes the family of nonempty, compact, convex subsets of $\mathbb{E}^n$.

In [4], the author showed that all isometries of $(\mathcal{C}(\mathbb{E}^n), d_{\mathrm{H}})$ are induced by the above-mentioned formula from the isometries of $\mathbb{E}^n$. In [2], it was shown that the same holds for the isometries of $(\mathfrak{C}(\mathbb{E}^n), d_{\mathrm{H}})$ and in [3] the authors generalized these observations to certain non-Euclidean cases. In 2005, in [1] the author generalized the results for Euclidean spaces to proper, uniquely geodesic, geodesically complete metric spaces, in which geodesics do not split.

Theorem 1 generalizes [1, Theorem 2] by omitting the assumption that the metric space we consider is proper, that is, that all of its closed balls are compact. Therefore, it applies to all strictly convex normed spaces with the metric generated by the norm, instead of only the proper, thus finite-dimensional ones.

The remainder of the paper is structured as follows: in Section 2, we recall basic definitions regarding metric spaces and geodesics. Then we explore the behaviour of the Hausdorff metric on the family of nonempty compact subsets. Moreover, we introduce the notion of midpoints and midpoint extensions. The proof of Theorem 1 is presented in Section 3.

# 2. PRELIMINARIES

## 2.1. METRIC SPACES

Let $(X,d)$ be a metric space. For $x \in X$ and $r \geq 0$, denote

$$B(x,r) := \left\{ y \in X \colon d(x,y) < r \right\}, \qquad \overline{B}(x,r) := \left\{ y \in X \colon d(x,y) \leq r \right\},$$
$$S(x,r) := \left\{ y \in X \colon d(x,y) = r \right\},$$

which will be called an open ball, a closed ball and a sphere, respectively, with a centre $x$ and a radius $r$. If all closed balls are compact, we say that the space $X$ is *proper*.

For $A \subseteq X$, $r \geq 0$ we define the $r-$hull of $A$ as:

$$H(A,r) := \left\{ x \in X \colon (\exists a \in A) \;\; d(a,x) \leq r \right\} = \bigcup_{a \in A} \overline{B}(a,r).$$

Let $(X,d)$, $(Y,\rho)$ be two metric spaces and let $i \colon X \to Y$ be such that $\rho(i(x), i(y)) = d(x,y)$ for all $x, y \in X$. If $i$ is surjective, we will call it an *isometry*; otherwise, we will call it an *isometric embedding*.

## 2.2. GEODESICS

Let $(X,d)$ be a metric space and let $I$ be an interval, that is, $I$ is of the form:

$$[a,b],\ [a,\infty),\ (-\infty,b],\ (-\infty,\infty),$$

where $a,b \in \mathbb{R}$ and $a \le b$. Isometric embedding $\gamma\colon I \to X$ shall be called a *geodesic*. In the case when $I = [a,b]$, we shall say that the geodesic $\gamma\colon I \to X$ *connects* $\gamma(a)$ *and* $\gamma(b)$. If $\gamma(a) = \gamma(b)$, $\gamma$ shall be called *trivial*. Let $\mathrm{Dom}(\gamma)$ and $\mathrm{Im}(\gamma)$ denote the domain and the image of $\gamma$, respectively.

We shall say that the metric space $(X,d)$ is *geodesic*, if for all $x,y \in X$ there exists a geodesic connecting $x$ with $y$. If for all $x,y \in X$ all geodesics connecting $x$ with $y$ have the same image, we shall say that $X$ is *uniquely geodesic*.

Geodesic metric space $(X,d)$ will be called *geodesically complete*, if for every non-trivial geodesic $\gamma$ there exists its *biinfinite extension*, that is, a geodesic $\widetilde{\gamma}\colon (-\infty,\infty) \to X$ such that $\widetilde{\gamma}|_{\mathrm{Dom}(\gamma)} = \gamma$. If for every non-trivial geodesic $\gamma$ in $X$ the image of every biinfinite extension of $\gamma$ is the same, we shall say that *geodesics do not split*.

## 2.3. SPACE OF NONEMPTY COMPACT SUBSETS

Let $(X,d)$ be a metric space. While the Hausdorff metric $d_{\mathrm{H}}$ is a metric on the family $\mathfrak{B}(X)$ of nonempty, closed and bounded subsets of $X$, we will work with the family $\mathfrak{C}(X)$ of nonempty compact subsets of $X$. The two families are different precisely in the case when $X$ is not a proper metric space, that is, when there are non-compact closed balls.

The next two lemmas allow us to find the value of $d_{\mathrm{H}}$ in some rather special cases.

**Lemma 2.** *Let $(X,d)$ be a metric space and let $K \in \mathfrak{C}(X)$. Let $k,g \in K$ be such that $d(k,g) =$* $\mathrm{diam}(K)$. *Then $d(K,\{g\}) =$* $\mathrm{diam}(K)$.

*Proof.* Notice that for all $\widetilde{k} \in K$ we have $d(g,\widetilde{k}) \le \mathrm{diam}(K)$. Therefore, since $g \in K$, we have $d_{\mathrm{H}}(K,\{g\}) \le \mathrm{diam}(K)$. On the other hand, for $r < \mathrm{diam}(K)$ we have $k \notin H(\{g\},r)$, so $K \nsubseteq H(\{g\},r)$. Thus, $d_{\mathrm{H}}(K,\{g\}) \ge \mathrm{diam}(K)$. Hence, $d(K,\{g\}) = \mathrm{diam}(K)$. □

**Lemma 3.** *Let $(X,d)$ be a metric space, $x \in X$ and $r > 0$. Let $K \in \mathfrak{C}(X)$ be such that $K \subseteq S(x,r)$. Suppose there are $k \in K$ and $s \in S(x,r)$ such that $d(k,s) = 2r$. Then $d_{\mathrm{H}}(K,\{s\}) = 2r$.*

*Proof.* First, notice that for $\lambda < 2r$ we have $k \notin H(\{s\},\lambda)$, so $K \nsubseteq H(\{s\},\lambda)$. Therefore, $d_{\mathrm{H}}(K,\{s\}) \ge 2r$. Now, notice that for all $\widetilde{k} \in K$ we have $d(\widetilde{k},s) \le d(\widetilde{k},x) + d(x,s) = 2r$, so $K \subseteq H(\{s\},2r)$. Also, $\{s\} \subseteq H(\{k\},2r) \subseteq H(K,2r)$, so $d_{\mathrm{H}}(K,\{s\}) \le 2r$. Thus, $d_{\mathrm{H}}(K,\{s\}) = 2r$. □

The following lemmas show that the Hausdorff metric $d_{\mathrm{H}}$ has fairly nice properties when we deal with the family $\mathfrak{C}(X)$.

**Lemma 4.** *Let $(X,d)$ be a metric space. Then, for $K,C \in \mathfrak{C}(X)$ we have:*

$$d_{\mathrm{H}}(K,C) := \min\left\{r \geq 0 \colon K \subseteq H(C,r) \;\;\wedge\;\; C \subseteq H(K,r)\right\}.$$

*Moreover, we have:*

$$\forall K,C \in \mathfrak{C}(X) \;\; \forall k \in K \;\; \exists c \in C \quad d(k,c) \leq d_{\mathrm{H}}(K,C).$$

*Proof.* Let $K,C \in \mathfrak{C}(X)$. Let us fix $k \in K$. For each $n \in \mathbb{N}$, there exists $c_n \in C$ such that $d(k,c_n) \leq d_{\mathrm{H}}(K,C) + 1/n$. Since $C$ is compact, we have a subsequence $(c_{n_m})_m$ and $c \in C$ such that $c_{n_m} \to c$ as $m \to \infty$. Passing to the limit on a subsequence we get $d(k,c) \leq d_{\mathrm{H}}(K,C)$, and the second claim follows.

Since $k \in K$ was arbitrary, we have $K \subseteq H(C, d_{\mathrm{H}}(K,C))$. In the same fashion, we get that $C \subseteq H(K, d_{\mathrm{H}}(K,C))$. This, by the definition of $d_{\mathrm{H}}$, proves the first claim. □

**Lemma 5.** *Let $(X,d)$ be a metric space and let $K,C \in \mathfrak{C}(X)$. Then at least one of the following two possibilities must occur:*

- $\exists k \in K \;\; \forall c \in C \quad d(k,c) \geq d_{\mathrm{H}}(K,C)$*;*
- $\exists c \in C \;\; \forall k \in K \quad d(k,c) \geq d_{\mathrm{H}}(K,C)$*.*

*Proof.* Suppose neither of the two possibilities occur. Set $K$ is compact, so the map $k \mapsto \min\left\{d(k,c) \colon c \in C\right\}$ reaches its maximum $M_1$ on $K$. Thus, $K \subseteq H(C, M_1)$. Similarly, the function $c \mapsto \min\left\{d(k,c) \colon k \in K\right\}$ reaches its maximum $M_2$ on $C$, so $C \subseteq H(K, M_2)$. Let $M := \max(M_1, M_2)$. By our assumption, $M < d_{\mathrm{H}}(K,C)$, which is in a contradiction with the definition of $d_{\mathrm{H}}(K,C)$. □

## 2.4. MIDPOINTS AND MIDPOINT EXTENSIONS

The following subsection is dedicated to the study of midpoints and midpoint extensions. The lemmas will be used extensively in the proof of the main result.

**Definition 6.** *Let $(X,d)$ be a metric space and let $x,z \in X$. Point $y \in X$ shall be called a* midpoint *between $x$ and $z$, if:*

$$d(x,y) = d(y,z) = \frac{1}{2} d(x,z).$$

*The set of all midpoints between $x$ and $z$ will be denoted by* $\mathrm{Mid}(x,z)$. *In the case when this set is a singleton, we shall denote its only element by* $\mathrm{mid}(x,z)$.

**Lemma 7.** *Let $(X,d)$ be a metric space and let $x,y,z \in X$. Suppose that*

$$d(x,y), d(y,z) \leq D \quad \text{and } d(x,z) \geq 2D$$

*for some $D \geq 0$. Then $y \in \mathrm{Mid}(x,z)$ and all the above inequalities are, in fact, equalities.*

*Proof.* By the triangle inequality:

$$2D \leq d(x,z) \leq d(x,y) + d(y,z) \leq D + D = 2D.$$

Therefore, all inequalities above are, in fact, equalities. In particular, we get that:

$$\frac{1}{2} d(x,z) = D = d(x,y) = d(y,z).$$

Thus, $y \in \mathrm{Mid}(x,z)$ and we have proved all the necessary equalities. □

**Corollary 8.** *Let $(X,d)$ be a metric space and let $x,z \in X$. Then*

$$\mathrm{Mid}(x,z) = \overline{B}\left(x, \frac{d(x,z)}{2}\right) \cap \overline{B}\left(z, \frac{d(x,z)}{2}\right).$$

*Proof.* Denote $D := \overline{B}\left(x, \frac{d(x,z)}{2}\right) \cap \overline{B}\left(z, \frac{d(x,z)}{2}\right)$. Inclusion $\mathrm{Mid}(x,z) \subseteq D$ follows from the definition of $\mathrm{Mid}(x,z)$. On the other hand, if $y \in D$, then $d(x,y), d(y,z) \leq \frac{d(x,z)}{2}$ and by Lemma 7 we get that $y \in \mathrm{Mid}(x,z)$. Therefore, $\mathrm{Mid}(x,z) \supseteq D$. □

**Lemma 9.** *Let $(X,d)$ be a geodesic metric space. Let $x,z \in X$ and $y \in \mathrm{Mid}(x,z)$. Let $\gamma_{xy}\colon [-a,0] \to X$ and $\gamma_{yz}\colon [0,a] \to X$ be geodesics connecting $x$ with $y$, and $y$ with $z$, respectively, where $a := d(x,y)$. Then $\gamma\colon [-a,a] \to X$ defined by the formula:*

$$\gamma(t) := \begin{cases} \gamma_{xy}(t), & \text{if } \ -a \leq t \leq 0, \\ \gamma_{yz}(t), & \text{if } \ 0 \leq t \leq a, \end{cases}$$

*is a geodesic connecting $x$ with $z$.*

*Proof.* First, let us notice that we only need to check that $d(\gamma(s),\gamma(t)) = |s-t|$ for $s \leq 0$ and $t \geq 0$. Thus, let $s \leq 0$ and $t \geq 0$.

Notice that:

$$\begin{aligned} d(\gamma(s),\gamma(t)) &\leq d(\gamma(s),\gamma(0)) + d(\gamma(0),\gamma(t)) \\ &= d(\gamma_{xy}(s),\gamma_{xy}(0)) + d(\gamma_{yz}(0),\gamma_{yz}(t)) = 0 - s + t - 0 = |s-t|. \end{aligned}$$

Therefore, $d(\gamma(s),\gamma(t)) \leq |s-t|$. On the other hand, we have:

$$\begin{aligned} 2a - d(\gamma(s),\gamma(t)) &= d(\gamma(-a),\gamma(a)) - d(\gamma(s),\gamma(t)) \leq d(\gamma(-a),\gamma(s)) + d(\gamma(t),\gamma(a)) \\ &= d(\gamma_{xy}(-a),\gamma_{xy}(s)) + d(\gamma_{yz}(t),\gamma_{yz}(a)) = a + s + a - t = 2a - |s-t|. \end{aligned}$$

Hence, $d(\gamma(s),\gamma(t)) \geq |s-t|$. Since we have proved inequalities in both directions, we have an equality, and, since $s \leq 0$, $t \geq 0$ were arbitrary, $\gamma$ is a geodesic connecting $\gamma(-a) = x$ with $\gamma(a) = z$. □

**Lemma 10.** *Let $(X,d)$ be a uniquely geodesic metric space. Then for all $x,z \in X$ there exists a unique midpoint between $x$ and $z$.*

*Proof.* First, notice that $\mathrm{Mid}(x,z) \neq \emptyset$. Indeed, since $X$ is geodesic, there is a geodesic $\gamma_1\colon [0,d(x,z)] \to X$ connecting $x$ with $z$ and $\gamma_1(\frac{d(x,z)}{2}) \in \mathrm{Mid}(x,z)$.

Next, let us fix $y,y' \in \mathrm{Mid}(x,z)$. Since $X$ is geodesic, there are geodesics $\gamma_{xy}, \gamma_{xy'}\colon [-a,0] \to X$ connecting $x$ with $y$ and $y'$, respectively, and geodesics $\gamma_{yz}, \gamma_{y'z}\colon [0,a] \to X$, connecting $y$ and $y'$ with $z$, respectively, where

$$a = d(x,y) = d(x,y') = d(y,z) = d(y',z).$$

Note that since $y \in \mathrm{Mid}(x,z)$, we also have $d(x,z) = 2a$.

Let us define functions $\gamma, \gamma'\colon [-a,a] \to X$ by the formulas:

$$\gamma(t) := \begin{cases} \gamma_{xy}(t), & \text{if } \ -a \leq t \leq 0, \\ \gamma_{yz}(t), & \text{if } \ 0 \leq t \leq a, \end{cases} \qquad \gamma'(t) := \begin{cases} \gamma_{xy'}(t), & \text{if } \ -a \leq t \leq 0, \\ \gamma_{y'z}(t), & \text{if } \ 0 \leq t \leq a. \end{cases}$$

By Lemma 9, $\gamma$ and $\gamma'$ are geodesics in $X$, connecting $x$ with $z$. Therefore, since $X$ is uniquely geodesic, images of these geodesics are equal.

Now, notice that for every $t \in [-a,a]$ we have:

$$d(x,\gamma(t)) = d(\gamma(-a),\gamma(t)) = t + a = d(\gamma'(-a),\gamma'(t)) = d(x,\gamma'(t)),$$

which means that each element of the image of these geodesics is uniquely determined by its distance from $x$. Thus, $\gamma(t) = \gamma'(t)$ for $t \in [-a,a]$ and, in particular, $y = \gamma(0) = \gamma'(0) = y'$. This shows that $\mathrm{Mid}(x,z)$ is a singleton. $\square$

**Lemma 11.** *Let $(X,d)$ be a uniquely geodesic metric space and let $x,z$ be its elements. Then $\{\mathrm{mid}(x,z)\} = \mathrm{mid}(\{x\},\{z\})$.*

*Proof.* One can easily verify that $\{\mathrm{mid}(x,z)\} \in \mathrm{Mid}(\{x\},\{z\})$ by calculating appropriate distances. Fix $K \in \mathrm{Mid}(\{x\},\{z\})$. Since $d_{\mathrm{H}}(\{x\},\{z\}) = d(x,z)$, we have:

$$K \subseteq H\left(\{x\}, \frac{d(x,z)}{2}\right) \cap H\left(\{z\}, \frac{d(x,z)}{2}\right) = \overline{B}\left(x, \frac{d(x,z)}{2}\right) \cap \overline{B}\left(z, \frac{d(x,z)}{2}\right) = \mathrm{Mid}(x,z),$$

where the last equality follows from Corollary 8. Thus, $K \subseteq \mathrm{Mid}(x,z)$ and, since $K \neq \emptyset$ as an element of $\mathfrak{C}(X)$ and $\mathrm{Mid}(x,z) = \{\mathrm{mid}(x,z)\}$ by Lemma 10, we have $K = \{\mathrm{mid}(x,z)\}$. $\square$

**Lemma 12.** *Let $(X,d)$ be a uniquely geodesic metric space. Let $K,C \in \mathfrak{C}(X)$ be such that there is a unique midpoint in $\mathfrak{C}(X)$ between them. Then* $\operatorname{dist}(K,C) = d_{\mathrm{H}}(K,C)$.

*Proof.* By the definition of the Hausdorff metric, we have $\operatorname{dist}(K,C) \leq d_{\mathrm{H}}(K,C)$. Suppose that $\operatorname{dist}(K,C) < d_{\mathrm{H}}(K,C)$. Then there exist $k \in K$ and $c \in C$ such that $d(k,c) < d_{\mathrm{H}}(K,C)$. Let $g := \operatorname{mid}(k,c)$. There exists $r > 0$ such that

$$\overline{B}(g,r) \subseteq \overline{B}(k,D) \cap \overline{B}(c,D) \quad \text{ and } \quad r < \frac{d(k,c)}{4},$$

where $D := \frac{1}{2} d_{\mathrm{H}}(K,C)$. Let $G$ be the only element of $\operatorname{Mid}(K,C)$. We have two possibilities: either $\overline{B}(g,r) \subseteq G$ or not.

In the first case, let $\gamma$ be a geodesic connecting $g$ with $k$. Denote $\widetilde{G} := G \setminus B(g, \frac{r}{4}) \cup \{g\}$ and $\widetilde{g} := \gamma(\frac{r}{8})$. Clearly, $\widetilde{G} \in \mathfrak{C}(X)$ and we have $\widetilde{g} \in G$ but $\widetilde{g} \notin \widetilde{G}$. We will show that $\widetilde{G} \in \operatorname{Mid}(K,C)$. Let us notice that $\widetilde{G} \subseteq G \subseteq H(K,D) \cap H(C,D)$.

Next, we will show that $H(\widetilde{G},D) = H(G,D)$. Since $\widetilde{G} \subseteq G$, we have $H(\widetilde{G},D) \subseteq H(G,D)$. Now, fix $x \in H(G,D)$ and suppose that $x \notin H(\widetilde{G},D)$. Then, there exists $g' \in B(g, \frac{r}{4})$ such that $d(g',x) \leq D$.

Notice that if $d(g',x) \leq \frac{r}{2}$, then

$$d(g,x) \leq d(g,g') + d(g',x) \leq \frac{r}{4} + \frac{r}{2} \leq r \leq D,$$

so, $d(g,x) \leq D$. Since $g \in \widetilde{G}$, this means that $x \in H(\widetilde{G},D)$, despite our assumption. Otherwise, let $\gamma'$ be a geodesic connecting $g'$ with $x$. Then $\gamma'(\frac{r}{2}) \in \overline{B}(g,r) \setminus B(g, \frac{r}{4}) \subseteq \widetilde{G}$ and $d(x, \gamma'(\frac{r}{2})) \leq D$. Indeed, we have:

$$\frac{r}{4} = \frac{r}{2} - \frac{r}{4} \leq d(\gamma'(\tfrac{r}{2}), g') - d(g',g) \leq d(\gamma'(\tfrac{r}{2}), g) \leq d(\gamma'(\tfrac{r}{2}), g') + d(g',g) \leq \frac{r}{2} + \frac{r}{4} \leq r,$$

and

$$d(x, \gamma'(\tfrac{r}{2})) = d(\gamma'(d(g',x)), \gamma'(\tfrac{r}{2})) = d(g',x) - \tfrac{r}{2} \leq D - \tfrac{r}{2} \leq D.$$

Thus, $x \in H(\widetilde{G},D)$, despite our assumption.

We see that in both cases $x \in H(\widetilde{G},D)$. Thus, $H(G,D) \subseteq H(\widetilde{G},D)$, so $H(G,D) = H(\widetilde{G},D)$. Therefore, we have $d_{\mathrm{H}}(K,\widetilde{G}), d_{\mathrm{H}}(C,\widetilde{G}) \leq D$, which, combined with $d_{\mathrm{H}}(K,C) = 2D$ and Lemma 7, gives us $\widetilde{G} \in \operatorname{Mid}(K,C)$. Since $\widetilde{g} \in G$ but $\widetilde{g} \notin \widetilde{G}$ and $\widetilde{G} \in \operatorname{Mid}(K,C)$, we have a contradiction with the uniqueness of the midpoint between $K$ and $C$.

If $\overline{B}(g,r) \not\subseteq G$, let $\widetilde{g} \in \overline{B}(g,r) \setminus G$ and $\widetilde{G} := G \cup \{\widetilde{g}\}$. Clearly, $\widetilde{G} \in \mathfrak{C}(X)$. Since $G \subseteq \widetilde{G}$, we have $K,C \subseteq H(G,D) \subseteq H(\widetilde{G},D)$. On the other hand, we have $G \subseteq H(K,D) \cap H(C,D)$ and $\overline{B}(g,r) \subseteq \overline{B}(k,D) \cap \overline{B}(c,D)$, which implies that $\widetilde{G} = G \cup \{\widetilde{g}\} \subseteq H(K,D) \cap H(C,D)$. Thus, $d_{\mathrm{H}}(K,\widetilde{G}), d_{\mathrm{H}}(C,\widetilde{G}) \leq D$, which, combined with $d_{\mathrm{H}}(K,C) = 2D$ and Lemma 7, implies that $\widetilde{G} \in \operatorname{Mid}(K,C)$. Since $\widetilde{g} \notin G$, but $\widetilde{g} \in \widetilde{G}$ and $\widetilde{G} \in \operatorname{Mid}(K,C)$, we have a contradiction with the uniqueness of the midpoint between $K$ and $C$.

Therefore, our assumption that $\operatorname{dist}(K,C) < d_{\mathrm{H}}(K,C)$ leads to a contradiction. Thus, $\operatorname{dist}(K,C) = d_{\mathrm{H}}(K,C)$. □

**Lemma 13.** *Let $(X,d)$ be a uniquely geodesic metric space and let $x \in X$, $r > 0$. Let $F \subseteq S(x,r)$ be an element of $\mathfrak{C}(X)$ such that there exists $F' \in \mathrm{Mid}(\{x\},F)$. Then $F'$ is the only element of* $\mathrm{Mid}(\{x\},F)$.

*Proof.* Denote $\widetilde{F} := \{\mathrm{mid}(x,f)\colon f \in F\}$. Note that $d_{\mathrm{H}}(\{x\},F) = r$, which implies that $d_{\mathrm{H}}(\{x\},F') = d_{\mathrm{H}}(F',F) = \frac{r}{2}$. We will show that $F' = \widetilde{F}$, which will prove the uniqueness as $\widetilde{F}$ does not depend on the choice of $F'$.

First, notice that since $F \subseteq S(x,r)$, we have $r = d(x,f)$ for all $f \in F$. Therefore, by Corollary 8, for all $f \in F$ we have $\{\mathrm{mid}(x,f)\} = \mathrm{Mid}(x,f) = \overline{B}\left(x,\frac{r}{2}\right) \cap \overline{B}\left(f,\frac{r}{2}\right)$, where the first equality follows from the uniqueness of midpoints in $X$. Therefore,

$$F' \subseteq H\left(\{x\},\frac{r}{2}\right) \cap H\left(F,\frac{r}{2}\right) = \overline{B}\left(x,\frac{r}{2}\right) \cap \bigcup_{f\in F} \overline{B}\left(f,\frac{r}{2}\right) = \bigcup_{f\in F} \overline{B}\left(x,\frac{r}{2}\right) \cap \overline{B}\left(f,\frac{r}{2}\right) = \widetilde{F}.$$

Hence, $F' \subseteq \widetilde{F}$.

Now, let us fix $\widetilde{f} \in \widetilde{F}$. There exists $f \in F$ such that $\widetilde{f} = \mathrm{mid}(x,f)$. By Lemma 4, there exists $f' \in F'$ such that $d(f',f) \leq d_{\mathrm{H}}(F',F) = \frac{r}{2}$. Since $F' \subseteq H(\{x\},\frac{r}{2}) = \overline{B}(x,\frac{r}{2})$, we have $d(x,f') \leq \frac{r}{2}$. Since $F \subseteq S(x,r)$, we have $d(x,f) = r$, which, combined with the previous inequalities and Lemma 7, gives us $f' \in \mathrm{Mid}(x,f)$. As midpoints are unique in $X$, we have $f' = \widetilde{f}$. Thus, $\widetilde{f} \in F'$ and, since $\widetilde{f}$ was arbitrary, $\widetilde{F} \subseteq F'$. □

**Lemma 14.** *Let $(X,d)$ be a uniquely geodesic metric space and let $x \in X$, $r > 0$. Let $F \subseteq S(x,r)$ be nonempty and finite. Then there exists a unique midpoint in $\mathfrak{C}(X)$ between $\{x\}$ and $F$.*

*Proof.* Consider the set $\widetilde{F} := \{\mathrm{mid}(x,f)\colon f \in F\}$. This set is nonempty and finite, so it is an element of $\mathfrak{C}(X)$. One can check that $r = d_{\mathrm{H}}(\{x\},F)$ and $d_{\mathrm{H}}(\{x\},\widetilde{F}), d_{\mathrm{H}}(\widetilde{F},F) = \frac{r}{2}$. Therefore, $\widetilde{F} \in \mathrm{Mid}(\{x\},F)$. Hence, by Lemma 13, the claim follows. □

**Lemma 15.** *Let $(X,d)$ be a metric space and let $i$ be an isometry on this space. Then:*

1. $\forall x,z \in X \quad i\left[\mathrm{Mid}(x,z)\right] = \mathrm{Mid}(i(x),i(z))$,
2. $\forall x,y,z \in X \quad y = \mathrm{mid}(x,z) \iff i(y) = \mathrm{mid}(i(x),i(z))$.

*Proof.* All statements follow from the definitions. □

**Definition 16.** *Let $(X,d)$ be a metric space and let $x,y \in X$. Point $z \in X$ will be called a* midpoint extension *by $x$ over $y$, if $y \in \mathrm{Mid}(x,z)$. The set of all midpoint extensions by $x$ over $y$ will be denoted by* $\mathrm{Mex}(x,y)$. *In the case when this set is a singleton, we shall denote its only element by* $\mathrm{mex}(x,z)$. *If $z \in \mathrm{Mex}(x,y)$ is such that $y = \mathrm{mid}(x,z)$, then $z$ will be called a* regular extension *(r.ex.) by $x$ over $y$.*

**Lemma 17.** *Let $(X,d)$ be a uniquely geodesic, geodesically complete metric space, in which geodesics do not split (UGUGC space for short). Then for every $x,y \in X$ there exists a unique midpoint extension by $x$ over $y$.*

*Proof.* Choose $x,y \in X$. First, notice that if $x = y$, then $x = \mathrm{mex}(x,y)$. Therefore, let us assume that $x \neq y$. Denote $a := d(x,y)$. Let $\gamma_{xy}\colon [-a,0] \to X$ be a geodesic connecting $x$ with $y$ and let $\widetilde{\gamma_{xy}}$ be its biinfinite extension. Then $z := \widetilde{\gamma_{xy}}(a)$ is a midpoint extension by $x$ over $y$.

Now, fix $z' \in \mathrm{Mex}(x,y)$. Let $\gamma_{yz'}\colon [0,a] \to X$ be a geodesic connecting $y$ with $z'$. Then, by Lemma 9, the function $\gamma\colon [-a,a] \to X$ defined by the formula:

$$\gamma(t) := \begin{cases} \gamma_{xy}(t), & \text{if } -a \leq t \leq 0, \\ \gamma_{yz'}(t), & \text{if } 0 \leq t \leq a \end{cases}$$

is a geodesic connecting $x$ with $z'$. Let $\widetilde{\gamma}$ be its biinfinite extension. Notice that it is also a biinfinite extension of $\gamma_{xy}$. Since geodesics in $X$ do not split, these extensions have the same image.

Now, notice that the map $\mathbb{R} \ni t \mapsto (|-a-t|,|t|) \in \mathbb{R}^2$ is injective and for every $t \in \mathbb{R}$ we have:

$$\begin{bmatrix} d(x,\widetilde{\gamma_{xy}}(t)) \\ d(y,\widetilde{\gamma_{xy}}(t)) \end{bmatrix} = \begin{bmatrix} d(\widetilde{\gamma_{xy}}(-a),\widetilde{\gamma_{xy}}(t)) \\ d(\widetilde{\gamma_{xy}}(0),\widetilde{\gamma_{xy}}(t)) \end{bmatrix} = \begin{bmatrix} |-a-t| \\ |t| \end{bmatrix} = \begin{bmatrix} d(\widetilde{\gamma}(-a),\widetilde{\gamma}(t)) \\ d(\widetilde{\gamma}(0),\widetilde{\gamma}(t)) \end{bmatrix} = \begin{bmatrix} d(x,\widetilde{\gamma}(t)) \\ d(y,\widetilde{\gamma}(t)) \end{bmatrix}.$$

This, combined with the fact that $\widetilde{\gamma_{xy}}$ and $\widetilde{\gamma}$ have the same image, means that each element of this image is uniquely determined by its distances from $x$ and $y$. Moreover, it also implies that $\widetilde{\gamma_{xy}}(t) = \widetilde{\gamma}(t)$ for all $t \in \mathbb{R}$, so, in particular,

$$z = \widetilde{\gamma_{xy}}(a) = \widetilde{\gamma}(a) = z'.$$

Therefore, since $z' \in \mathrm{Mex}(x,y)$ was arbitrary, the set $\mathrm{Mex}(x,y)$ is a singleton. □

**Corollary 18.** *Let $(X,d)$ be a UGUGC space. Let $x \in X$, $r > 0$ and $y \in S(x,r)$. Then $\mathrm{mex}(y,x)$ is the only element of $\overline{B}(x,r)$, which is at least $2r$ away from $y$.*

*Proof.* Denote $y' = \mathrm{mex}(y,x)$. Let $y'' \in \overline{B}(x,r)$ be such that $d(y,y'') \geq 2r$. Then, by Lemma 7, $x \in \mathrm{Mid}(y,y'')$, so $y'' \in \mathrm{Mex}(y,x)$. Thus, as midpoint extensions in $X$ are unique, $y' = y''$. □

**Lemma 19.** *Let $(X,d)$ be a UGUGC space. Then for every $x,y \in X$ the singleton $\{\mathrm{mex}(x,y)\}$ is an r.ex. in $\mathfrak{C}(X)$ by $\{x\}$ over $\{y\}$.*

*Proof.* Since $\mathrm{mid}(x,\mathrm{mex}(x,y)) = y$, by Lemma 11, we have:

$$\mathrm{Mid}(\{x\},\{\mathrm{mex}(x,y)\}) = \left\{\{\mathrm{mid}(x,\mathrm{mex}(x,y))\}\right\} = \{\{y\}\}.$$

Thus, $\{\mathrm{mex}(x,y)\}$ is an r.ex. in $\mathfrak{C}(X)$ by $\{x\}$ over $\{y\}$. □

**Lemma 20.** *Let $(X,d)$ be a UGUGC space. Let $K,G \in \mathfrak{C}(X)$ be such that there is an r.ex. in $\mathfrak{C}(X)$ by $K$ over $G$. Then*

$$\forall g \in G \ \ \exists! k \in K \quad d(k,g) \leq d_{\mathrm{H}}(K,G).$$

*Moreover, the above inequality is, in fact, an equality.*

*Proof.* Let us notice that by Lemma 4, for all $g \in G$ there exists $k \in K$ such that $d(k,g) \leq d_{\mathrm{H}}(K,G)$. We need to prove the uniqueness. Suppose there exist $g \in G$ and $k_1,k_2 \in K$ such that $d(k_i,g) \leq d_{\mathrm{H}}(K,G)$ for $i \in \{1,2\}$. Let $C \in \mathfrak{C}(X)$ be r.ex. by $K$ over $G$. Then there exists $c \in C$ such that $d(g,c) \leq d_{\mathrm{H}}(G,C) = d_{\mathrm{H}}(K,G)$. By the definition of $C$ we have $G = \mathrm{mid}(K,C)$ so, by Lemma 12, we have $\mathrm{dist}(K,C) = d_{\mathrm{H}}(K,C) = 2d_{\mathrm{H}}(K,G)$. Thus, $d(k_i,c) \geq 2d_{\mathrm{H}}(K,G)$ for $i \in \{1,2\}$.

By Lemma 7, the inequalities $d(g,c), d(k_i,g) \leq d_{\mathrm{H}}(K,G)$ and $d(k_i,c) \geq 2d_{\mathrm{H}}(K,G)$ for $i \in \{1,2\}$ give us that $g \in \mathrm{Mid}(k_i,c)$ and $d(k_i,g) = d_{\mathrm{H}}(K,G)$ for $i \in \{1,2\}$. Thus, $k_1,k_2 \in \mathrm{Mex}(c,g)$. Therefore, by the uniqueness of the midpoint extensions in $X$, we have $k_1 = k_2$. This proves the uniqueness, and we have the equality $d(k_1,g) = d_{\mathrm{H}}(K,G)$ as needed. □

**Lemma 21.** *Let $(X,d)$ be a UGUGC space. Let $K,C \in \mathfrak{C}(X)$ be such that there is a singleton $\{x\} \in \mathrm{Mid}(K,C)$. Then, at least one of the sets $K$, $C$ is a singleton.*

*Proof.* Denote $D := d_{\mathrm{H}}(K,\{x\})$. By Lemma 5, at least one of the following two statements is true:

- $\exists k \in K \ \ \forall c \in C \quad d(k,c) \geq d_{\mathrm{H}}(K,C) = 2D$;
- $\exists c \in C \ \ \forall k \in K \quad d(k,c) \geq d_{\mathrm{H}}(K,C) = 2D$.

Suppose the first possibility occurs with $k \in K$ as above. Since $K,C \subseteq H(\{x\},D) = \overline{B}(x,D)$, by Lemma 7, for every $c \in C$ we have $x \in \mathrm{Mid}(k,c)$, or, equivalently, $c \in \mathrm{Mex}(k,x)$. Since midpoint extensions in $X$ are unique, $C$ is a singleton. If the latter possibility occurs, we can analogously prove that the set $K$ is a singleton. □

**Lemma 22.** *Let $(X,d)$ be a metric space and let $i$ be an isometry of this space. Then:*

1. $\forall x,y \in X \quad i\left[\mathrm{Mex}(x,y)\right] = \mathrm{Mex}(i(x),i(y))$;
2. $\forall x,y,z \in X \quad z = \mathrm{mex}(x,y) \iff i(z) = \mathrm{mex}(i(x),i(y))$;
3. $\forall x,y,z \in X \quad$ *$z$ is an r.ex. by $x$ over $y$ $\iff$ $i(z)$ is an r.ex. by $i(x)$ over $i(y)$.*

*Proof.* All statements follow directly from the definitions and Lemma 15. □

# 3. PROOF OF THE MAIN RESULT

The proof of the main result is divided into three steps:

1. We prove that $K \in \mathfrak{C}(X)$ is a singleton if and only if $I(K)$ is a singleton,
2. We show that if, additionally, we have $I(\{x\}) = \{x\}$ for all $x \in X$, then $I(F) = F$, where $F$ is a finite subset of a sphere,
3. We prove that for arbitrary isometry $I$ of $(\mathfrak{C}(X), d_{\mathrm{H}})$ there exists an isometry $i$ of $(X,d)$ such that $I(K) = i[K]$ for all $K \in \mathfrak{C}(X)$.

The first step, established in Lemma 23, plays the key role. Indeed, it allows us to construct an isometry $i$ of $(X,d)$ such that $I(K) = i[K]$ for all singletons $K$. In the last step, we show that this equality is in fact satisfied for all $K \in \mathfrak{C}(X)$.

**Lemma 23.** *Let $(X,d)$ be a UGUGC space. Let $I$ be an isometry of $(\mathfrak{C}(X), d_{\mathrm{H}})$. Then $I\left[\mathfrak{S}(X)\right] = \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ denotes the family of all singletons of $X$.*

*Proof.* We shall prove that $I\left[\mathfrak{C}(X) \setminus \mathfrak{S}(X)\right] \subseteq \mathfrak{C}(X) \setminus \mathfrak{S}(X)$. Suppose that this inclusion is not satisfied. Then, there exists $K \in \mathfrak{C}(X) \setminus \mathfrak{S}(X)$ such that $I(K) \in \mathfrak{S}(X)$.

There exist $k, g \in K$ such that $d(k,g) = \operatorname{diam}(K)$. Denote $D := d(k,g)$. By Lemma 2, we have $D = d_{\mathrm{H}}(K, \{g\})$. Let $c := \operatorname{mex}(k,g)$. Then $D = d(c,g) = d_{\mathrm{H}}(\{c\}, \{g\})$ and $2D = d(k,c)$. Therefore, $k \notin H(\{c\}, r)$ for $r < 2D$, so $K \not\subseteq H(\{c\}, r)$ for $r < 2D$. Thus, $2D \le d_{\mathrm{H}}(K, \{c\})$, so, by Lemma 7, $\{g\} \in \operatorname{Mid}(K, \{c\})$.

Since, by assumption, $I(K) \in \mathfrak{S}(X)$, there exists $k' \in X$ such that $I(K) = \left\{k'\right\}$. Fix $g' \in I(\{g\})$ and let $c'' := \operatorname{mex}(k', g')$. Since $D = d_{\mathrm{H}}(K, \{g\}) = d_{\mathrm{H}}(\left\{k'\right\}, I(\{g\}))$, we have $d(k', g') \le D$ and because of that, $d(g', c'') \le D$. Denote $C'' := I(\{c\}) \cup \left\{c''\right\}$.

Since $d(g', c'') \le D$ and $d_{\mathrm{H}}(I(\{g\}), I(\{c\})) = D$, we have $d_{\mathrm{H}}(I(\{g\}), C'') \le D$. Also, since $d_{\mathrm{H}}(\left\{k'\right\}, I(\{c\})) = 2D$, we have $d_{\mathrm{H}}(\left\{k'\right\}, C'') \ge 2D$. Note that $D = d_{\mathrm{H}}(\left\{k'\right\}, I(\{g\}))$, so, by Lemma 7, we have $I(\{g\}) \in \operatorname{Mid}(\left\{k'\right\}, C'')$. Denote $\widetilde{C} := I^{-1}(C'')$. By Lemma 15, we have $\{g\} \in \operatorname{Mid}(K, \widetilde{C})$. Therefore, since $K \notin \mathfrak{S}(X)$, by Lemma 21, we have $\widetilde{C} \in \mathfrak{S}(X)$.

Note that $X$ satisfies the assumptions of Lemma 19. Therefore, in $\mathfrak{C}(X)$ there are r.ex.'s by $\{c\}$ over $\{g\}$ and by $\widetilde{C}$ over $\{g\}$. Thus, by Lemma 22, there are r.ex.'s by $I(\{c\})$ over $I(\{g\})$ and by $C''$ over $I(\{g\})$. Hence, by Lemma 20, for each of the sets $I(\{c\})$, $C''$ there exist unique $c' \in I(\{c\})$ and $\widetilde{c}'' \in C''$ such that: $d(g', c'), d(g', \widetilde{c}'') \le D$. Since $\operatorname{mex}(k', g') = c'' \in C''$ and $d(g', c'') \le D$, we have $\widetilde{c}'' = c''$. Moreover, since $I(\{c\}) \subseteq C'' = I(\{c\}) \cup \left\{c''\right\}$, we have $c' = \widetilde{c}'' = c'' = \operatorname{mex}(k', g')$. This proves that for every $g' \in I(\{g\})$ we have $\operatorname{mex}(k', g') \in I(\{c\})$.

Furthermore, Lemma 20 also states that $d(g', c') = D$. Therefore, $d(g', \operatorname{mex}(k', g')) = D$ for every $g' \in I(\{g\})$ and, because of that, also $d(k', g') = D$ for every $g' \in I(\{g\})$. This shows that $I(\{g\}) \subseteq S(k', D)$.

Now, fix $\widetilde{c}' \in I(\{c\})$. By Lemma 4, there exists $\widetilde{g}' \in I(\{g\})$ such that $d(\widetilde{g}', \widetilde{c}') \leq D = d_{\mathrm{H}}(I(\{g\}), I(\{c\}))$. We have proved that $\mathrm{mex}(k', \widetilde{g}')$ is the only element of $I(\{c\})$ such that $d(\widetilde{g}', \mathrm{mex}(k', \widetilde{g}')) \leq D$ and we have $d(\widetilde{g}', \mathrm{mex}(k', \widetilde{g}')) = D$. Thus, $\widetilde{c}' = \mathrm{mex}(k', \widetilde{g}')$ and $d(\widetilde{g}', \widetilde{c}') = D$. This implies that $d(k', \widetilde{c}') = 2d(\widetilde{g}', \widetilde{c}') = 2D$. Since $\widetilde{c}' \in I(\{c\})$ is arbitrary, we have $I(\{c\}) \subseteq S(k', 2D)$.

We found that $I(\{g\}) \subseteq S(k', D)$ and $I(\{c\}) \subseteq S(k', 2D)$. Furthermore, by Lemma 15, since $\{g\} \in \mathrm{Mid}(K, \{c\})$, we have $I(\{g\}) \in \mathrm{Mid}(\{k'\}, I(\{c\}))$. Therefore, by Lemma 13, we know that $\mathrm{Mid}(\{k'\}, I(\{c\})) = \{I(\{g\})\}$ and thus $\mathrm{Mid}(K, \{c\}) = \{\{g\}\}$ by Lemma 15. Hence, by Lemma 12, we have:

$$2D = d_{\mathrm{H}}(K, \{c\}) \leq d(g, c) = D,$$

which implies that $D \leq 0$. However, since $D = d(k, g) = \mathrm{diam}(K)$ and $K$ is not a singleton, we have $D > 0$, a contradiction.

Hence, we have $I\left[\mathfrak{C}(X) \setminus \mathfrak{S}(X)\right] \subseteq \mathfrak{C}(X) \setminus \mathfrak{S}(X)$. Since $I$ is a bijection, the above inclusion implies that $I\left[\mathfrak{S}(X)\right] \supseteq \mathfrak{S}(X)$. Because $I^{-1}$ also is an isometry, we get $I^{-1}\left[\mathfrak{S}(X)\right] \supseteq \mathfrak{S}(X)$. Thus, we obtain $\mathfrak{S}(X) = I\left[I^{-1}\left[\mathfrak{S}(X)\right]\right] \supseteq I\left[\mathfrak{S}(X)\right]$. We have proved inclusions in both directions and $I\left[\mathfrak{S}(X)\right] = \mathfrak{S}(X)$ as needed. □

**Remark 24.** *Let $(X, d)$ be a UGUGC space. Let $I$ be an isometry of $(\mathfrak{C}(X), d_{\mathrm{H}})$. Since $I[\mathfrak{S}(X)] = \mathfrak{S}(X)$, the function $i\colon X \to X$ defined by:*

$$\forall x \in X \quad \{i(x)\} = I(\{x\})$$

*is an isometry of $(X, d)$. Therefore, map $J\colon \mathfrak{C}(X) \to \mathfrak{C}(X)$ defined as follows:*

$$\forall K \in \mathfrak{C}(X) \quad J(K) := i^{-1}\left[I(K)\right]$$

*is an isometry of $(\mathfrak{C}(X), d_{\mathrm{H}})$ and $J(\{x\}) = \{x\}$ for every $x \in X$.*

**Lemma 25.** *Let $(X, d)$ be a UGUGC space. Let $I$ be an isometry of $(\mathfrak{C}(X), d_{\mathrm{H}})$ such that $I(\{y\}) = \{y\}$ for every $y \in X$. Let $x \in X$ and $r > 0$. Then $I(F) = F$, where $F$ is a finite subset of $S(x, r)$.*

*Proof.* Denote $F' := I(F)$. First, let us notice that $d_{\mathrm{H}}(\{x\}, F) = r$ and therefore $d_{\mathrm{H}}(\{x\}, F') = d_{\mathrm{H}}(I(\{x\}), I(F)) = r$. This shows that $F' \subseteq \overline{B}(x, r)$. Furthermore, since, by Lemma 14, there is a unique midpoint in $\mathfrak{C}(X)$ between $F$ and $\{x\}$, Lemma 15 states that there must also be a unique midpoint between $F' = I(F)$ and $\{x\} = I(\{x\})$. Therefore, by Lemma 12, $\mathrm{dist}(x, F') = r$ which, combined with $F' \subseteq \overline{B}(x, r)$, gives us $F' \subseteq S(x, r)$.

Let us fix $f \in F$ and let $\widetilde{f} := \mathrm{mex}(f, x)$. Then $d(x, \widetilde{f}) = r$ as $d(x, f) = r$. We have $d(f, \widetilde{f}) = 2r$ so, by Lemma 3, $d_{\mathrm{H}}(F, \{\widetilde{f}\}) = 2r$. Thus, $d_{\mathrm{H}}(F', \{\widetilde{f}\}) = d_{\mathrm{H}}(I(F), I(\{\widetilde{f}\})) = 2r$. Therefore, there exists $a \in F'$ such that $d(a, \widetilde{f}) = 2r$. Since, by Corollary 18, $f$ is the only element of $\overline{B}(x, r)$ such that $d(f, \widetilde{f}) = 2r$ and $F' \subseteq \overline{B}(x, r)$, we have $f = a \in F'$. Since $f \in F$ was arbitrary, we have $F \subseteq F'$.

Next, let $f' \in F'$ and let $\widetilde{f'} := \operatorname{mex}(f',x)$. Then $d(x,\widetilde{f'}) = r$. We have $d(f',\widetilde{f'}) = 2r$ and $F' \subseteq S(x,r)$, which, by Lemma 3, implies that $d_{\mathrm{H}}(F',\{\widetilde{f'}\}) = 2r$. Therefore,

$$d_{\mathrm{H}}(F,\{\widetilde{f'}\}) = d_{\mathrm{H}}(I(F),I(\{\widetilde{f'}\})) = d_{\mathrm{H}}(F',\{\widetilde{f'}\}) = 2r,$$

so $d_{\mathrm{H}}(F,\{\widetilde{f'}\}) = 2r$. Therefore, there exists $b \in F$ such that $d(b,\{\widetilde{f'}\}) = 2r$. Since $F \subseteq \overline{B}(x,r)$ and, by Corollary 18, $f'$ is the only element of $\overline{B}(x,r)$ such that $d(f',\widetilde{f'}) = 2r$, we have $f' = b \in F$. Since $f' \in F'$ was arbitrary, we have proved that $F' \subseteq F$.

Since we have proved inclusions in both directions, we have $F = F' = I(F)$ as needed. □

*Proof of Theorem 1.* Since $I$ is an isometry of $(\mathfrak{C}(X),d_{\mathrm{H}})$, by Remark 24, we know that there exists an isometry $i\colon X \to X$ such that for all $x \in X$ we have $i[\{x\}] = I(\{x\})$. Remark 24 also states that the function $J\colon \mathfrak{C}(X) \to \mathfrak{C}(X)$ defined by the formula $J(\cdot) = i^{-1}[I(\cdot)]$ is an isometry of $(\mathfrak{C}(X),d_{\mathrm{H}})$ such that for all $x \in X$ we have $J(\{x\}) = \{x\}$. Note that if we prove that $J(K) = K$ for all $K \in \mathfrak{C}(X)$, then $I(K) = i[K]$ for all $K \in \mathfrak{C}(X)$ as needed.

Let us fix $K \in \mathfrak{C}(X)$. Suppose that $J(K) \neq K$. Then $K \not\subseteq J(K)$ or $K \not\supseteq J(K)$. Let us first consider the former possibility, that is, suppose that $K \not\subseteq J(K)$. Thus, there exists $k \in K \setminus J(K)$. Since $J(K)$ is compact, there exists $k' \in J(K)$ such that $d(k,k') = \operatorname{dist}(k,J(K)) > 0$. Denote $\mu := \frac{1}{2}d(k,k')$.

Since $k \notin J(K)$, the family $\{B(\widetilde{k},\frac{1}{2}d(k,\widetilde{k}))\colon \widetilde{k} \in J(K)\}$ is an open cover of $J(K)$. Since $J(K)$ is compact, there exist $k'_1, \ldots, k'_n \in J(K)$ such that $\{B(k'_i,\frac{1}{2}d(k,k'_i))\colon i \in \{1,\ldots,n\}\}$ is a finite subcover of $J(K)$.

Take $\lambda > 2\mu + \operatorname{diam}(J(K))$ and fix $i \in \{1,\ldots,n\}$. First, we shall prove that $d(k,k'_i) < \lambda$. Indeed, we have:

$$d(k,k'_i) \leq d(k,k') + d(k',k'_i) = 2\mu + d(k',k'_i) \leq 2\mu + \operatorname{diam}(J(K)) < \lambda.$$

There exists a geodesic $\gamma_{kk'_i}\colon \left[0,d(k,k'_i)\right] \to X$ connecting $k$ with $k'_i$. It has a biinfinite extension $\widetilde{\gamma_{kk'_i}}$. Let $c_i := \widetilde{\gamma_{kk'_i}}(\lambda)$. Since, by definition, $\widetilde{\gamma_{kk'_i}}|_{\mathrm{Dom}(\gamma_{kk'_i})} = \gamma_{kk'_i}$, we have $\widetilde{\gamma_{kk'_i}}(0) = k$ and $\widetilde{\gamma_{kk'_i}}(d(k,k'_i)) = k'_i$. Therefore, since $d(k,k'_i) < \lambda$ and $\widetilde{\gamma_{kk'_i}}$ is a geodesic, we have:

$$\lambda = d(k,c_i) = d(k,k'_i) + d(k'_i,c_i).$$

Denote $S := \{c_1, \ldots, c_n\}$. Since $d(k,c_i) = \lambda$ for all $i \in \{1,\ldots,n\}$, we have $S \subseteq S(k,\lambda)$. This shows that for $r < \lambda$ we have $\{k\} \not\subseteq H(S,r)$ and, since $\{k\} \subseteq K$, also $K \not\subseteq H(S,r)$. Therefore, $d_{\mathrm{H}}(K,S) \geq \lambda$.

Fix $i \in \{1,\ldots,n\}$. We have $B(k'_i,\frac{1}{2}d(k,k'_i)) \subseteq B(c_i,\lambda-\mu)$. Indeed, since

$$\frac{1}{2}d(k,k'_i) + d(k'_i,c_i) = d(k,k'_i) + d(k'_i,c_i) - \frac{1}{2}d(k,k'_i) = \lambda - \frac{1}{2}d(k,k'_i) \leq \lambda - \mu,$$

we have:

$$B\left(k'_i,\frac{1}{2}d(k,k'_i)\right) \subseteq B\left(c_i,\frac{1}{2}d(k,k'_i) + d(k'_i,c_i)\right) \subseteq B(c_i,\lambda-\mu).$$

Since $\{B(k_i', \frac{1}{2}d(k,k_i')) \colon i \in \{1,\ldots,n\}\}$ is a finite subcover of $J(K)$, we have:

$$J(K) \subseteq \bigcup_{i=1}^{n} B\left(k_i', \frac{1}{2}d(k,k_i')\right) \subseteq \bigcup_{i=1}^{n} B(c_i, \lambda-\mu) \subseteq \bigcup_{i=1}^{n} \overline{B}(c_i, \lambda-\mu) = H(S, \lambda-\mu).$$

Next, since for all $i \in \{1,\ldots,n\}$ we have $d(k,k_i') \geq 2\mu$, the equations

$$\lambda = d(k,c_i) = d(k,k_i') + d(k_i', c_i)$$

tell us that $d(k_i', c_i) \leq \lambda - 2\mu$. Therefore, as all $k_i'$ are elements of $J(K)$, we have $S \subseteq H(J(K), \lambda - 2\mu)$. Hence, by the definition of the Hausdorff metric, $d_{\mathrm{H}}(S, J(K)) \leq \lambda - \mu$.

Since $S \subseteq S(k,\lambda)$ is finite, by Lemma 25 we get $J(S) = S$. We arrive at a contradiction, because

$$\lambda \leq d_{\mathrm{H}}(S,K) = d_{\mathrm{H}}(J(S), J(K)) = d_{\mathrm{H}}(S, J(K)) \leq \lambda - \mu$$

implies that $\lambda \leq \lambda - \mu$ which is impossible, since $\mu > 0$. Therefore, $K \subseteq J(K)$.

We have proved that if $J$ is an isometry such that $J(\{x\}) = \{x\}$ for all $x \in X$, then for all $K \in \mathfrak{C}(X)$ we have $K \subseteq J(K)$. Notice that the isometry $J^{-1}$ also has the property that $J^{-1}(\{x\}) = \{x\}$ for all $x \in X$. Therefore, we can use the above approach to the set $J(K) \in \mathfrak{C}(X)$ to prove that

$$J(K) \subseteq J^{-1}\big(J(K)\big) = K.$$

Therefore, $K \subseteq J(K)$ and $J(K) \subseteq K$, so $K = J(K)$. Since $K \in \mathfrak{C}(X)$ was arbitrary, we have:

$$\forall K \in \mathfrak{C}(X) \quad J(K) = K,$$

which, as we mentioned at the beginning, ends the proof. □

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# CHARACTERIZATION OF THE RANGE OF THE FRACTIONAL INTEGRAL OPERATOR IN $L^2$ SPACE

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**Abstract:** The aim of this paper is to provide the characterization of the range of the fractional integral operator in $L^2$ space. It is well known that this range is an interpolation space of order $\alpha$ between $L^2$ and ${}_0H^1$ spaces. We characterize explicitly this interpolation space in terms of the subspace of the Sobolev-Slobodeckij space. This enables us to define the domain of Riemann-Liouville derivative in terms of Sobolev spaces.
**Keywords:** Sobolev-Slobodeckij space, fractional integral operator, complex interpolation space
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## 1. INTRODUCTION

The reason for writing this paper is the studying of differential equations with fractional order derivative. Fractional calculus has its origins in the 19th century. Abel, Liouville and Riemann introduced the concept of Riemann-Liouville derivative. However, solving fractional differential equations with this derivative requires defining the initial condition of the fractional order. Nevertheless, in 1967 the Caputo derivative was proposed and this fractional derivative can be applied to model various physical phenomena, because the initial condition can be given as a derivative of integer order.

What makes fractional calculus so interesting is the fact that fractional derivatives are non local operators. Thanks to this property, they are able to model the memory effects e.g. in fractional Stefan problem (see e.g. [5], [14], [15] and [16]). An equally interesting area of research deals with inverse problems in which we want for example to determine the order of fractional derivative which characterizes the diffusion. It is interesting from an application point of view, because very often it is difficult to measure this parameter directly (see e.g. [4] and [8]).

To prove the existence and uniqueness of the solution of some fractional differential equation, first we have to define the domain of an appropriate fractional operator. It is necessary in order to define the space in which we want to find the solution of the equation.

In order to better explain our motivation, we recall that the fractional integral operator $I^{\alpha}$ defined on $L^2(0,T)$ is given by the formula

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad \text{for} \quad f \in L^2(0,T), \quad \mathrm{Re}\alpha > 0. \tag{1}$$

Moreover, by $\partial^{\alpha} := \frac{d}{dt} I^{1-\alpha}$ we denote the Riemann-Liouville derivative and by $D^{\alpha}$ we denote the fractional Caputo derivative given by the formula $D^{\alpha}u(t) := \partial^{\alpha}[u(\cdot) - u(0)](t)$. We are especially interested in the research of the time fractional diffusion equation

$$\partial^{\alpha}(u - u_0) - \mathrm{div}(A(t,x)Du) = f, \quad t \in (0,T),\ x \in \Omega, \tag{2}$$

where $A = (a_{i,j})$ is $R^{n \times n}$ valued.

It is proved that $I^{\alpha}(L^2(0,T)) = [L^2(0,T),\ {}_0H^1(0,T)]_{\alpha}$ (see [6]). Moreover, we know that

$$\partial^{\alpha} : [L^2(0,T),\ {}_0H^1(0,T)]_{\alpha} \to L^2(0,T)$$

is an isomorphism, and $\partial^{\alpha} I^{\alpha} f = f$ for $f \in L^2(0,T)$, $I^{\alpha}\partial^{\alpha} f = f$ for $f \in [L^2(0,T),\ {}_0H^1(0,T)]_{\alpha}$. From the above considerations, we understand that the characterization of the image of fractional integral operator $I^{\alpha}$ is necessary in order to define the domain of the Riemann-Liouville derivative.

The purpose of this paper is to provide the characterization of the interpolation space $[L^2(0,T),\ {}_0H^1(0,T)]_{\alpha}$ in terms of some subspace of Sobolev-Slobodeckij space. It is a well known result, but it is difficult to find a complete proof in literature. This is the reason why the authors decided to provide a detailed and direct proof of this result. We were inspired by Theorem 11.6 and remark 11.5 in [9]. However, in this paper we present an alternative proof of Theorem 11.6 and remark 11.5 from [9].

The analysis of that proof is the first step to characterize $I^{\alpha}(L^p(0,T))$ for $p \neq 2$. However, in this case we will need more sophisticated spaces in order to describe the interpolation space $[L^p(0,T),\ {}_0W^{1,p}(0,T)]_{\alpha}$. We suppose that it would require using Bessel potential spaces.

Our paper is divided into seven sections. In Section 2, we present some basic definitions. In particular, we recall the definition of the complex interpolation space and we define some fractional Sobolev spaces which are very important for our further considerations. Then, in Section 3, we formulate the main result. In Section 4, we recall the definition and some useful properties of the strongly continuous semigroup. Moreover, we find the infintesimal generator of the translation semigroup. Finally, we will formulate another interesting characterization of the complex interpolation space. In Section 5, we are going to prove the Hardy's inequality. Furthermore, we will present some imbedding results. In Section 6, we will prove the main result of the paper. In Section 7, the reader can find an appendix in which we collect some basic results that are used in the article.

# 2. NOTATION

Let us recall some definitions which will be very useful in our further considerations.

**Definition 1** (Chapter 11, Section 11.1 in [11])**.** *Let $X_0$, $X_1$ be a couple of Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Then $(X_0,X_1)$ is an* ***interpolation pair*** *if there exists a Hausdorff vectorial topological space $X$ such that $X_i \hookrightarrow X$ for $i=0,1$.*

**Definition 2** (Def. 11.1.2 in [11] - Complex interpolation method)**.** *Let $(X_0,X_1)$ be an interpolation pair. By $\mathcal{F}(X_0,X_1)$ we denote the set of functions $f: S=\{z\in\mathbb{C}: 0\le Rez\le 1\}\to X_0+X_1$ such that*

1. *$f$ is continuous and bounded in $S$,*
2. *$f$ is analytic in $S^0=\{z\in\mathbb{C}: 0<Rez<1\}$,*
3. *$f(it)\in X_0$ and $f(it+1)\in X_1$ for all $t\in\mathbb{R}$,*
4. *functions $t\mapsto f(it)$ and $t\mapsto f(it+1)$ are bounded and continuous with respect to the spaces $X_0$ and $X_1$, respectively.*

*We provide the space $\mathcal{F}(X_0,X_1)$ with the norm*

$$\|f\|_{\mathcal{F}(X_0,X_1)}=\max\{\sup_{t\in\mathbb{R}}\|f(it)\|_{X_0},\sup_{t\in\mathbb{R}}\|f(1+it)\|_{X_1}\}.$$

*We define the space*

$$[X_0,X_1]_\theta=\{f(\theta): f\in\mathcal{F}(X_0,X_1)\}$$

*with the norm*

$$\|\phi\|_\theta=\inf\{\|f\|_{\mathcal{F}(X_0,X_1)}: f(\theta)=\phi\}.$$

**Definition 3** (Def. 1.3.2.1 in [3])**.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $1\le p<+\infty$. We denote by $W^{s,p}(\Omega)$ the space of all distributions $u$ defined in $\Omega$, such that*

1. *$D^\alpha u\in L^p(\Omega)$, for $|\alpha|\le m$, when $s=m$ is a nonnegative integer,*
2. *$u\in W^{m,p}(\Omega)$ and*
$$\int_\Omega\int_\Omega \frac{|D^\alpha u(x)-D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}dxdy<+\infty$$
*for $|\alpha|=m$, when $s=m+\sigma$ is nonnegative and is not an integer.*

*We define a Banach norm on $W^{s,p}(\Omega)$ by*

$$\|u\|_{W^{m,p}(\Omega)}=\left\{\sum_{|\alpha|\le m}\int_\Omega |D^\alpha u|^p dx\right\}^{\frac{1}{p}}$$

*in the case 1., and by*

$$\|u\|_{W^{s,p}(\Omega)} = \left\{ \|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}} dxdy \right\}^{\frac{1}{p}}$$

*in the case 2.*

**Definition 4.** *Let $T > 0$ and $\alpha \in (0,1)$. By $H^\alpha(0,T)$ we denote the space*

$$H^\alpha(0,T) := W^{\alpha,2}(0,T)$$

*and provide it with the norm*

$$\|u\|_{H^\alpha(0,T)} = \left\{ \|u\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|u(x)-u(y)|^2}{|x-y|^{1+2\alpha}} dxdy \right\}^{\frac{1}{2}}.$$

**Definition 5.** *Let $\Omega = (0,T)$. We define*

$${}_0H^1(\Omega) = \{u : u \in H^1(\Omega),\ u(0) = 0\}.$$

# 3. THE MAIN RESULT

Our purpose is to show the following theorem:

**Theorem 6.** *Let $T > 0$. Then, for all $\alpha \in (0,1)$, we have*

$$[L^2(0,T),\ {}_0H^1(0,T)]_\alpha = {}_0H^\alpha(0,T),$$

*where*

$${}_0H^\alpha(0,T) = \begin{cases} H^\alpha(0,T) & \text{for } \alpha \in (0, \frac{1}{2}), \\ \{H^{\frac{1}{2}}(0,T) : \int_0^T \frac{|u(t)|^2}{t} dt < \infty\} & \text{for } \alpha = \frac{1}{2}, \\ \{H^\alpha(0,T) : u(0) = 0\} & \text{for } \alpha \in (\frac{1}{2}, 1), \end{cases}$$

*and the norm is defined by*

$$\|u\|_{{}_0H^\alpha(0,T)} = \begin{cases} \|u\|_{H^\alpha(0,T)} & \text{for } \alpha \neq \frac{1}{2}, \\ \left( \|u\|^2_{H^{\frac{1}{2}}(0,T)} + \int_0^T \frac{|u(t)|^2}{t} dt \right)^{\frac{1}{2}} & \text{for } \alpha = \frac{1}{2}, \end{cases}$$

*where*

$$\|u\|_{H^\alpha(0,T)} = \left\{ \|u\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{|u(x)-u(y)|^2}{|x-y|^{1+2\alpha}} dxdy \right\}^{\frac{1}{2}}.$$

# 4. SEMIGROUPS AND THEIR PROPERTIES

**Definition 7** (Def. 1.1 in [13]). *Let $X$ be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from $X$ into $X$ is a semigroup of bounded linear operators on $X$ if*

1. *$T(0) = I$ ($I$ is the identity operator on $X$),*
2. *$T(t+s) = T(t)T(s)$ for every $t,s \geq 0$ (the semigroup property).*

**Definition 8** (Definition 2.1 in [13]). *A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on $X$ is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t\downarrow 0} T(t)x = x \quad \text{for every} \quad x \in X.$$

**Definition 9** (Section 1.1 in [13]). *The linear operator $A$ defined by*

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists in } X \right\}$$

*and*

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \quad \text{for} \quad x \in D(A)$$

*is the infinitesimal generator of the semigroup $T(t)$, where $D(A)$ is the domain of $A$.*

**Proposition 10** (Section I.5.a in [1]). *For every strongly continuous semigroup $(T(t))_{t\geq 0}$, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \tag{3}$$

*for all $t \geq 0$.*

**Definition 11** (Section I.5.a in [1]). *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ we call*

$$\omega_0 := \inf\{\omega \in \mathbb{R} : \exists\, M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t}\ \forall\, t \geq 0\}$$

*its growth bound. Moreover, a semigroup is called bounded if we can take $\omega = 0$ in (3), and contractive if $\omega = 0$ and $M = 1$ is possible.*

**Theorem 12** (Thr. 1.10 in [1]). *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space $X$ and take constants $\omega \in \mathbb{R}$, $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}$$

*for $t \geq 0$. For the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$, the following properties hold.*

1. *If $\lambda \in \mathbb{C}$ such that $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)xds$ exists for all $x \in X$ , then $\lambda \in \rho(A)$ and $R(\lambda,A) = R(\lambda)$.*

2. *If $Re\lambda > \omega$, then $\lambda \in \rho(A)$, and the resolvent is given by the integral expression in 1.*

3. *$\|R(\lambda,A)\| \leq \frac{M}{Re\lambda-\omega}$ for all $Re\lambda > \omega$.*

**Definition 13.** *Let $-\infty < a < b < +\infty$. On the Banach space $L^p(a,b)$, $1 \leq p \leq \infty$, we define the right translation semigroup by*

$$(T_r(t)f)(x) := \begin{cases} f(x-t) & \text{if } x-t > a, \\ 0 & \text{if } x-t < a \end{cases}$$

*for all $t \geq 0$.*

**Remark 1.** *We observe that the one parameter family $\{T_r(t)\}_{t\geq 0}$ defined in Definition 13 is actually a semigroup. To this purpose, we show that $\{T_r(t)\}_{t\geq 0}$ satisfy the conditions of Definition 7. Let $f \in L^p(a,b)$ and $1 \leq p \leq \infty$. We have*
*1.*

$$(T_r(0)f)(x) = \begin{cases} f(x) & \text{if } x > a, \\ 0 & \text{if } x < a, \end{cases}$$

*but we always take $x \in (a,b)$, so $(T_r(0)f)(x) = f(x)$ for $x \in (a,b)$. Hence, $T(0) = I$.*
*2. Let $t,s \geq 0$. We get*

$$\begin{aligned}(T_r(t)T_r(s)f)(x) &= \begin{cases} (T_r(s)f)(x-t) & \text{if } x-t > a, \\ 0 & \text{if } x-t < a \end{cases} \\ &= \begin{cases} f(x-t-s) & \text{if } x-t-s > a, \\ 0 & \text{if } x-t-s < a \end{cases} \\ &= (T_r(t+s)f)(x).\end{aligned}$$

*Thus, $\{T_r(t)\}_{t\geq 0}$ is really a semigroup.*

**Remark 2.** *Let $(T_r(t))_{t\geq 0}$ be the right translation semigroup. Then*

$$T_r(t) = 0$$

*for all $t \geq b-a$.*

**Remark 3.** *The right translation semigroup $(T_r(t))_{t\geq 0}$ defined on the Banach space $L^p(a,b)$, $1 \leq p \leq \infty$, is a contractive semigroup, as shown below. We see that:*

*(i) if $t \in [0,b-a)$, then we have*

$$\begin{aligned}\|T_r(t)f\|^p_{L^p(a,b)} &= \int_a^b |(T_r(t)f)(x)|^p dx = \int_{a+t}^b |f(x-t)|^p dx \\ &= \int_a^{b-t} |f(s)|^p ds \leq \int_a^b |f(s)|^p ds = \|f\|^p_{L^p(a,b)}\end{aligned}$$

*for* $1 \le p < \infty$ *and*

$$\begin{aligned}\|T_r(t)f\|_{L^\infty(a,b)} &= \operatorname*{ess\,sup}_{x\in(a,b)} |T_r(t)f(x)| = \operatorname*{ess\,sup}_{x\in(a+t,b)} |f(x-t)| \\ &= \operatorname*{ess\,sup}_{s\in(a,b-t)} |f(s)| \le \operatorname*{ess\,sup}_{s\in(a,b)} |f(s)| = \|f\|_{L^\infty(a,b)}\end{aligned}$$

*for* $p = \infty$,

*(ii) if* $t \ge b-a$, *then we have*

$$\|T_r(t)f\|_{L^p(a,b)} = 0 \le \|f\|_{L^p(a,b)},$$

*for* $1 \le p \le \infty$.

*Hence, for* $1 \le p \le \infty$ *we get*

$$\|T_r(t)\|_{L^p(a,b)} \le 1 \quad \textit{for all} \quad t \ge 0.$$

**Proposition 14** (Section II.2.b in [1])**.** *The generator of the (right) translation semigroup* $(T_r(t))_{t\ge 0}$ *on the space* $X := L^p(a,b)$, $1 \le p < \infty$, *is given by*

$$Af := -f'$$

*with domain:*

$$D(A) = \{f \in L^p(a,b) : f \textit{ absolutely continuous and } f' \in L^p(a,b)\}.$$

*Proof.* Let $B : D(B) \to L^p(a,b)$ be the infinitesimal generator of the semigroup $(T_r(t))_{t\ge 0}$ according to the Definition 9. We want to show that $B = A$.
1. In the first step, we will show that $B \subset A$. We take $f \in D(B)$. From Definition 9 we know that

$$Bf = \lim_{t\downarrow 0} \frac{T_r(t)f - f}{t} \text{ in } L^p(a,b). \tag{4}$$

Let $c,d \in (a,b)$. We have $L^p(a,b) \hookrightarrow L^p(c,d) \hookrightarrow L^1(c,d)$, and hence

$$\begin{aligned}\left| \int_c^d \frac{T_r(t)f(x) - f(x)}{t} dx - \int_c^d Bf(x)dx \right| &\le \int_c^d \left| \frac{T_r(t)f(x) - f(x)}{t} - Bf(x) \right| dx \\ &= \left\| \frac{T_r(t)f - f}{t} - Bf \right\|_{L^1(c,d)} \\ &\le C \left\| \frac{T_r(t)f - f}{t} - Bf \right\|_{L^p(a,b)}.\end{aligned}$$

From (4) we know that the right hand side of the above inequality converges to 0 as $t \downarrow 0$. Thus, we have

$$\int_c^d \frac{T_r(t)f(x) - f(x)}{t} dx \overset{t\to 0^+}{\to} \int_c^d Bf(x)dx. \tag{5}$$

We have $t \to 0^+$, so we can assume that $t$ is so small that $c-t>a$. Due to this observation, we can write

$$\begin{aligned}\int_c^d \frac{T_r(t)f(x)-f(x)}{t}dx &= \frac{1}{t}\int_c^d T_r(t)f(x)dx - \frac{1}{t}\int_c^d f(x)dx \\ &= \frac{1}{t}\int_{c-t}^{d-t} f(x)dx - \frac{1}{t}\int_c^d f(x)dx = -\frac{1}{t}\int_{d-t}^{d} f(x)dx + \frac{1}{t}\int_{c-t}^{c} f(x)dx.\end{aligned}$$

Using the Lebesgue Differentiation Theorem, we get

$$-\frac{1}{t}\int_{d-t}^{d} f(x)dx + \frac{1}{t}\int_{c-t}^{c} f(x)dx \overset{t\to 0^+}{\to} -f(d)+f(c) \quad \text{for a.e. } c,d \in (a,b). \tag{6}$$

From (5) and (6) we have

$$f(d) = f(c) + \int_c^d (-Bf)(x)dx \quad \text{for a.e. } c,d \in (a,b).$$

We set $c_0 \in (a,b)$ such that

$$f(d) = f(c_0) + \int_{c_0}^d (-Bf)(x)dx \quad \text{for a.e. } d \in (a,b).$$

Let

$$\tilde{f}(d) := f(c_0) + \int_{c_0}^d (-Bf)(x)dx \quad \text{for all } d \in (a,b).$$

Then we have $f = \tilde{f}$ a.e. in $(a,b)$ and $\tilde{f}(c_0) = f(c_0)$. Thus,

$$\tilde{f}(d) = \tilde{f}(c_0) + \int_{c_0}^d (-B\tilde{f})(x)dx \quad \text{for all } d \in (a,b).$$

If we take $d_1, d_2 \in (a,b)$, then we have

$$\tilde{f}(d_1) = \tilde{f}(c_0) + \int_{c_0}^{d_1} (-B\tilde{f})(x)dx$$

and

$$\tilde{f}(d_2) = \tilde{f}(c_0) + \int_{c_0}^{d_2} (-B\tilde{f})(x)dx.$$

Hence, we get

$$\tilde{f}(d_2) = \tilde{f}(d_1) + \int_{d_1}^{d_2} (-B\tilde{f})(x)dx \quad \text{for all } d_1, d_2 \in (a,b).$$

Thus, according to the Theorem 28, $\tilde{f}$ is an absolutely continuous function with derivative (almost everywhere) equal to $-B\tilde{f} \in L^p(a,b)$. Thus, we have

$$D(B) \subset D(A) \quad \text{and} \quad A\big|_{D(B)} = B. \tag{7}$$

2. In the second step, we will deduce that $B = A$. To this purpose we make the following observations:
(i) Translation semigroup is a contractive semigroup, so

$$\|T_r(t)\|_{L^p(a,b)} \le Me^{\omega t}$$

for all $t \ge 0$ with $M = 1$ and $\omega = 0$. Hence, from Theorem 12, we obtain that $1 \in \rho(B)$.
(ii) We will also show below that $1 \in \rho(A)$. We know that

$$1 \in \rho(A) \Leftrightarrow \text{there exists the bounded operator } (A-I)^{-1} \text{ on } L^p(a,b).$$

We can see that for $f \in L^p(a,b)$

$$(A-I)^{-1}f = u \Leftrightarrow f = (A-I)u \Leftrightarrow f = -u' - u,$$

where $u \in D(A)$. Thus, $1 \in \rho(A)$ if and only if for all $f \in L^p(a,b)$ there exists a unique solution of the following equation:

$$f = -u' - u \tag{8}$$

and this solution belongs to $D(A)$. It is easy to see that the solution of (8) is given by

$$u(t) = -\int_a^t e^{s-t} f(s)ds.$$

Thus, we have

$$((A-I)^{-1}f)(t) = -\int_a^t e^{s-t} f(s)ds = -\int_a^t e^{-(t-s)} f(s)ds = -(f * e^{-s})(t),$$

and from Young's convolution inequality we get

$$\|(A-I)^{-1}f\|_{L^p(a,b)} = \|f * e^{-s}\|_{L^p(a,b)} \le \|e^{-s}\|_{L^1(a,b)} \|f\|_{L^p(a,b)} = |e^{-b} - e^{-a}| \|f\|_{L^p(a,b)}.$$

Hence, we have $1 \in \rho(A)$.
(iii) Due to (7) and observation (i), we obtain

$$(I-A)(D(B)) = (I-B)(D(B)) = L^p(a,b).$$

Moreover, using observation (ii) we get

$$D(A) = (I-A)^{-1}(L^p(a,b)).$$

Hence, we get

$$D(A) = (I-A)^{-1}(I-A)(D(B)) = D(B),$$

and then $A = B$. □

Now, we introduce some facts which will be necessary to formulate another characterization of complex interpolation space.
Let $X$ and $Y$ be two Hilbert spaces which we assume to be separable with

$$X \subset Y,\ X \text{ dense in } Y \text{ with continuous injection.} \tag{9}$$

Let $G(t)$ be a continuous semigroup on $Y$, that is:

$$\begin{cases} G(t) \in \mathbf{B}(Y,Y) \quad \forall\, t \geq 0, \quad G(0) = I, \\ \forall y \in Y, \quad \|G(t)y - y\|_Y \to 0 \quad \text{as } t \downarrow 0, \\ G(t)G(s) = G(t+s) \quad \forall t,s \geq 0. \end{cases} \tag{10}$$

Let $A$ be the infinitesimal generator of $G(t)$, with domain $D(A)$, which is a Hilbert space for the norm of the graph $(\|y\|_Y^2 + \|Ay\|_Y^2)^{\frac{1}{2}}$. We assume that

$$D(A) = X \quad \text{(with equivalent norms)} \tag{11}$$

We also assume that

$$\exists\, C > 0\, \forall\, t \geq 0 \quad \|G(t)\|_{\mathbf{B}(Y,Y)} \leq C. \tag{12}$$

**Theorem 15** (Chapter 1, Thr. 10.1 in [9])**.** *Let $X,Y$ satisfy* (9) *and let $G(t)_{t\geq 0}$ be any semigroup which satisfies* (10), (11) *and* (12)*. For $\theta \in (0,1)$, the following three statements are equivalent:*

$$a \in [X,Y]_\theta, \tag{13}$$

$$\begin{cases} \textit{There exists a function } u \textit{ which satisfies:} \\ a = u(0), \quad t^\alpha u \in L^2(0,\infty;X), \\ t^\alpha \frac{du}{dt} \in L^2(0,\infty;Y), \quad \theta = \frac{1}{2} + \alpha, \end{cases} \tag{14}$$

$$t^{\alpha-1}(G(t)a - a) \in L^2(0,\infty;Y). \tag{15}$$

*Furthermore, the norms*

$$\|a\|_{[X,Y]_\theta} \quad \textit{and} \quad \left(\|a\|_Y^2 + \int_0^\infty t^{2(\alpha-1)} \|G(t)a - a\|_Y^2 dt\right)^{\frac{1}{2}}$$

*are equivalent.*

# 5. AUXILIARY LEMMAS

In this chapter we will present the proof of the famous Hardy's inequality. In Section 5.2. we will show that the space of smooth functions with compact support is dense in $H^\alpha(0,+\infty)$ for $0 < \alpha \leq \frac{1}{2}$. Moreover, in Theorem 23 we show an important imbedding result, which we then use in the proof of Theorem 6.

## 5.1. HARDY'S INEQUALITY

Now, we formulate the convenient form of Hardy's inequality, which will be necessary for us in the proof of Theorem 23.

**Definition 16** (Chapter 1, Section 1.4.4 in [3]). *Let $1 \leq p \leq +\infty$ and $\alpha \in \mathbb{R}$. By $L^{p,\alpha}(0,+\infty)$ we denote the space of all measurable functions $u$ defined in $(0,+\infty)$ such that*

$$\|u\|^p_{L^{p,\alpha}(0,+\infty)} = \int_0^\infty |u(t)t^\alpha|^p dt < +\infty$$

*in the case when $1 \leq p < +\infty$, and*

$$\|u\|_{L^{\infty,\alpha}(0,+\infty)} = \operatorname*{ess\,sup}_{t>0} |t^\alpha u(t)| < +\infty$$

*in the case when $p = +\infty$.*

**Theorem 17** ([Chapter 1, Section 1.4.4 in [3] - Hardy's inequality). *Let $1 \leq p \leq +\infty$. We define two linear operators $H$ and $L$ by*

$$(Hu)(t) = \frac{1}{t}\int_0^t u(s)ds,$$

$$(Lu)(t) = \frac{1}{t}\int_t^\infty u(s)ds.$$

*If $1 \leq p \leq +\infty$, then $H$ is linear and continuous in $L^{p,\alpha}(0,+\infty)$ if and only if $\alpha + \frac{1}{p} < 1$, while $L$ is linear and continuous in $L^{p,\alpha}(0,+\infty)$ if and only if $\alpha + \frac{1}{p} > 1$. In both cases the norm of the operator is bounded by $|\alpha + \frac{1}{p} - 1|^{-1}$.*

*Proof.* 1. **Case $\mathbf{1 \leq p < +\infty}$.**
(i) We will show that $H$ is continuous in $L^{p,\alpha}(0,+\infty)$ if and only if $\alpha + \frac{1}{p} < 1$.

- First, we assume that $\alpha + \frac{1}{p} < 1$ and we want to show that the operator $H$ is continuous in $L^{p,\alpha}(0,+\infty)$, that is

$$\left\|\frac{1}{t}\int_0^t u(s)ds\right\|_{L^{p,\alpha}(0,+\infty)} \leq C\|u\|_{L^{p,\alpha}(0,+\infty)}. \tag{16}$$

We perform some calculations:

$$\left\|\frac{1}{t}\int_0^t u(s)ds\right\|^p_{L^{p,\alpha}(0,+\infty)} = \left\|t^\alpha\left(\frac{1}{t}\int_0^t u(s)ds\right)\right\|^p_{L^p(0,+\infty)} = \int_0^\infty \left|t^\alpha\left(\frac{1}{t}\int_0^t u(s)ds\right)\right|^p dt$$

$$\overset{s=e^y}{=} \int_0^\infty \left|t^\alpha\left(\frac{1}{t}\int_{-\infty}^{\ln t} u(e^y)e^y dy\right)\right|^p dt$$

$$
\begin{aligned}
&\overset{t=e^x}{=} \int_{\mathbb{R}} \left| e^{x\alpha} \left( e^{-x} \int_{-\infty}^{x} u(e^y) e^y dy \right) \right|^p e^x dx \\
&= \int_{\mathbb{R}} \left| e^{x\alpha} e^{-x} e^{\frac{x}{p}} \int_{-\infty}^{x} u(e^y) e^y dy \right|^p dx \\
&= \int_{\mathbb{R}} \left| e^{x(\alpha+\frac{1}{p}-1)} \int_{-\infty}^{x} u(e^y) e^y dy \right|^p dx \\
&= \int_{\mathbb{R}} \left| \int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} e^{y(\alpha+\frac{1}{p}-1)} u(e^y) e^y dy \right|^p dx \\
&= \int_{\mathbb{R}} \left| \int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} e^{y(\alpha+\frac{1}{p})} u(e^y) dy \right|^p dx \\
&\overset{\tilde{u}(y):=e^{y(\alpha+\frac{1}{p})}u(e^y)}{=} \left\| \int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})}^p .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|u\|_{L^{p,\alpha}(0,+\infty)}^p = \|t^\alpha u\|_{L^p(0,+\infty)}^p = \int_0^\infty |t^\alpha u(t)|^p dt &\overset{t=e^x}{=} \int_{\mathbb{R}} |e^{x\alpha} u(e^x)|^p e^x dx \\
&= \int_{\mathbb{R}} |e^{x(\alpha+\frac{1}{p})} u(e^x)|^p dx = \int_{\mathbb{R}} |\tilde{u}(x)|^p dx = \|\tilde{u}\|_{L^p(\mathbb{R})}^p .
\end{aligned}
$$

Thus, inequality (16) is equivalent to

$$
\left\| \int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})} \leq C \|\tilde{u}\|_{L^p(\mathbb{R})}. \tag{17}
$$

Now, we show (17). We notice that

$$
\int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy = (E * \tilde{u})(x),
$$

where

$$
E(x) = \begin{cases} e^{(\alpha+\frac{1}{p}-1)x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}
$$

Furthermore,

$$
\|E\|_{L^1(\mathbb{R})} = \int_0^\infty e^{(\alpha+\frac{1}{p}-1)x} dx = \frac{1}{1-\alpha-\frac{1}{p}.}
$$

Finally, from Young's inequality for convolutions we get

$$
\begin{aligned}
\left\| \int_{-\infty}^{x} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})} &= \|E * \tilde{u}\|_{L^p(\mathbb{R})} \\
&\leq \|E\|_{L^1(\mathbb{R})} \|\tilde{u}\|_{L^p(\mathbb{R})} = \frac{1}{1-\alpha-\frac{1}{p}} \|\tilde{u}\|_{L^p(\mathbb{R})}.
\end{aligned}
$$

Thus, we have (17), and hence (16).

- We want to show that if $H$ is continuous in $L^{p,\alpha}(0,+\infty)$, then $\alpha+\frac{1}{p}<1$. We prove it by contradiction. We assume that $\alpha+\frac{1}{p}\geq 1$. We will show that there exists $u\in L^{p,\alpha}(0,+\infty)$ such that the following inequality fails:

$$\left\|\frac{1}{t}\int_0^t u(s)ds\right\|_{L^{p,\alpha}(0,+\infty)} \leq C\|u\|_{L^{p,\alpha}(0,+\infty)}. \tag{18}$$

We take $u_R(t)=t^{-(\alpha+\frac{1}{p})}\chi_{(\frac{1}{R},R)}(t)$ for some $R>0$. Then

$$\|u_R\|^p_{L^{p,\alpha}(0,+\infty)}=\int_{\frac{1}{R}}^R t^{\alpha p}|u_R(t)|^p dt=\int_{\frac{1}{R}}^R t^{\alpha p}t^{-\alpha p-1}dt=2\ln R$$

and for $\alpha+\frac{1}{p}>1$ we have

$$\begin{aligned}
\|Hu_R\|^p_{L^{p,\alpha}(0,+\infty)} &= \int_0^\infty t^{\alpha p}|Hu_R(t)|^p dt=\int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_0^t u_R(s)ds\right|^p dt\\
&= \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_0^t s^{-\alpha-\frac{1}{p}}\chi_{(\frac{1}{R},R)}(s)ds\right|^p dt\\
&= \int_{\frac{1}{R}}^R t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^t s^{-\alpha-\frac{1}{p}}ds\right|^p dt+\int_R^\infty t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt\\
&= \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\int_{\frac{1}{R}}^t s^{-\alpha-\frac{1}{p}}ds\right|^p dt+\int_R^\infty t^{p(\alpha-1)}\left|\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt\\
&= \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Bigg|_{s=\frac{1}{R}}^{s=t}\right|^p dt\\
&\quad+\int_R^\infty t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Bigg|_{s=\frac{1}{R}}^{s=R}\right|^p dt\\
&= \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Bigg|_{s=\frac{1}{R}}^{s=t}\right|^p dt\\
&\quad+\int_R^\infty t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Bigg|_{s=\frac{1}{R}}^{s=R}\right|^p dt\\
&= \int_{\frac{1}{R}}^R \frac{1}{|-\alpha-\frac{1}{p}+1|^p}t^{p(\alpha-1)}|R^{\alpha+\frac{1}{p}-1}-t^{-\alpha-\frac{1}{p}+1}|^p dt\\
&\quad+\int_R^\infty \frac{1}{|-\alpha-\frac{1}{p}+1|^p}t^{p(\alpha-1)}|R^{\alpha+\frac{1}{p}-1}-R^{-\alpha-\frac{1}{p}+1}|^p dt\\
&= \int_{\frac{1}{R}}^R \frac{1}{|-\alpha-\frac{1}{p}+1|^p}t^{p(\alpha-1)}|R^{\alpha+\frac{1}{p}-1}-t^{-\alpha-\frac{1}{p}+1}|^p dt\\
&\quad+\frac{1}{|-\alpha-\frac{1}{p}+1|^p}|R^{\alpha+\frac{1}{p}-1}-R^{-\alpha-\frac{1}{p}+1}|^p\int_R^\infty t^{p(\alpha-1)}dt
\end{aligned}$$

$$= +\infty,$$

because $p(\alpha-1) > -1$.
If $\alpha + \frac{1}{p} = 1$, then

$$\begin{aligned}
\|Hu_R\|^p_{L^{p,\alpha}(0,+\infty)} &= \int_0^\infty t^{\alpha p}|Hu_R(t)|^p dt = \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_0^t u_R(s)ds\right|^p dt \\
&= \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_0^t s^{-\alpha-\frac{1}{p}}\chi_{(\frac{1}{R},R)}(s)ds\right|^p dt \\
&= \int_{\frac{1}{R}}^R t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^t s^{-\alpha-\frac{1}{p}}ds\right|^p dt + \int_R^\infty t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt \\
&= \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\int_{\frac{1}{R}}^t s^{-\alpha-\frac{1}{p}}ds\right|^p dt + \int_R^\infty t^{p(\alpha-1)}\left|\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt \\
&= \int_{\frac{1}{R}}^R t^{-1}\left|\int_{\frac{1}{R}}^t s^{-1}ds\right|^p dt + \int_R^\infty t^{-1}\left|\int_{\frac{1}{R}}^R s^{-1}ds\right|^p dt \\
&= \int_{\frac{1}{R}}^R t^{-1}(\ln t + \ln R)^p dt + (2\ln R)^p\int_R^\infty t^{-1}dt = +\infty.
\end{aligned}$$

Thus, inequality (18) fails for $u_R$.

(ii) We will show that $L$ is linear and continuous in $L^{p,\alpha}(0,+\infty)$ if and only if $\alpha + \frac{1}{p} > 1$.

- We want to show that if $\alpha + \frac{1}{p} > 1$, then the operator $L$ is continuous in $L^{p,\alpha}(0,+\infty)$. It means that we are going to show

$$\left\|\frac{1}{t}\int_t^\infty u(s)ds\right\|_{L^{p,\alpha}(0,+\infty)} \leq C\|u\|_{L^{p,\alpha}(0,+\infty)}. \tag{19}$$

We perform similar calculations as for the operator $H$:

$$\begin{aligned}
\left\|\frac{1}{t}\int_t^\infty u(s)ds\right\|^p_{L^{p,\alpha}(0,+\infty)} &= \left\|t^\alpha\left(\frac{1}{t}\int_t^\infty u(s)ds\right)\right\|^p_{L^p(0,+\infty)} \\
&= \int_0^\infty \left|t^\alpha\left(\frac{1}{t}\int_t^\infty u(s)ds\right)\right|^p dt \\
&\overset{s=e^y}{=} \int_0^\infty \left|t^\alpha\left(\frac{1}{t}\int_{\ln t}^\infty u(e^y)e^y dy\right)\right|^p dt \\
&\overset{t=e^x}{=} \int_{\mathbb{R}} \left|e^{x\alpha}\left(e^{-x}\int_x^\infty u(e^y)e^y dy\right)\right|^p e^x dx \\
&= \int_{\mathbb{R}} \left|e^{x\alpha}e^{-x}e^{\frac{x}{p}}\int_x^\infty u(e^y)e^y dy\right|^p dx \\
&= \int_{\mathbb{R}} \left|e^{x(\alpha+\frac{1}{p}-1)}\int_x^\infty u(e^y)e^y dy\right|^p dx
\end{aligned}$$

$$= \int_{\mathbb{R}} \left| \int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} e^{y(\alpha+\frac{1}{p}-1)} u(e^y) e^y dy \right|^p dx$$
$$= \int_{\mathbb{R}} \left| \int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} e^{y(\alpha+\frac{1}{p})} u(e^y) dy \right|^p dx$$
$$\overset{\tilde{u}(y):=e^{y(\alpha+\frac{1}{p})}u(e^y)}{=} \left\| \int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})}^p.$$

Furthermore,

$$\|u\|_{L^{p,\alpha}(0,+\infty)}^p = \|t^\alpha u\|_{L^p(0,+\infty)}^p = \int_0^\infty |t^\alpha u(t)|^p dt \overset{t=e^x}{=} \int_{\mathbb{R}} |e^{x\alpha} u(e^x)|^p e^x dx$$
$$= \int_{\mathbb{R}} |e^{x(\alpha+\frac{1}{p})} u(e^x)|^p dx = \int_{\mathbb{R}} |\tilde{u}(x)|^p dx = \|\tilde{u}\|_{L^p(\mathbb{R})}^p.$$

Thus, inequality (19) is equivalent to

$$\left\| \int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})} \leq C \|\tilde{u}\|_{L^p(\mathbb{R})}. \tag{20}$$

Now, we show (20). We notice that

$$\int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy = (E * \tilde{u})(x),$$

where

$$E(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ e^{(\alpha+\frac{1}{p}-1)x} & \text{if } x < 0. \end{cases}$$

Next,

$$\|E\|_{L^1(\mathbb{R})} = \int_{-\infty}^0 e^{(\alpha+\frac{1}{p}-1)x} dx = \frac{1}{\alpha + \frac{1}{p} - 1.}$$

Finally, from Young's convolution inequality we get

$$\left\| \int_x^{\infty} e^{(x-y)(\alpha+\frac{1}{p}-1)} \tilde{u}(y) dy \right\|_{L^p(\mathbb{R})} = \|E * \tilde{u}\|_{L^p(\mathbb{R})}$$
$$\leq \|E\|_{L^1(\mathbb{R})} \|\tilde{u}\|_{L^p(\mathbb{R})} = \frac{1}{\alpha + \frac{1}{p} - 1} \|\tilde{u}\|_{L^p(\mathbb{R})}.$$

Thus, we have (20), and hence (19).

- We want to show that if $L$ is continuous in $L^{p,\alpha}(0,+\infty)$, then $\alpha + \frac{1}{p} > 1$. We prove it by contradiction. We assume that $\alpha + \frac{1}{p} \leq 1$. We will show that there exists $u \in L^{p,\alpha}(0,+\infty)$ such that the following inequality fails:

$$\left\| \frac{1}{t} \int_t^{\infty} u(s) ds \right\|_{L^{p,\alpha}(0,+\infty)} \leq C \|u\|_{L^{p,\alpha}(0,+\infty)}. \tag{21}$$

We take $u_R(t) = t^{-(\alpha+\frac{1}{p})}\chi_{(\frac{1}{R},R)}(t)$. Then

$$\|u_R\|^p_{L^{p,\alpha}(0,+\infty)} = \int_{\frac{1}{R}}^{R} t^{\alpha p}|u_R(t)|^p dt = \int_{\frac{1}{R}}^{R} t^{\alpha p} t^{-\alpha p-1} dt = 2\ln R,$$

and for $\alpha + \frac{1}{p} < 1$ we get

$$\begin{aligned}
\|Lu_R\|^p_{L^{p,\alpha}(0,+\infty)} &= \int_0^\infty t^{\alpha p}|Lu_R(t)|^p dt = \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_t^\infty u_R(s)ds\right|^p dt \\
&= \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_t^\infty s^{-\alpha-\frac{1}{p}}\chi_{(\frac{1}{R},R)}(s)ds\right|^p dt \\
&= \int_0^{\frac{1}{R}} t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt + \int_{\frac{1}{R}}^R t^{\alpha p}\left|\frac{1}{t}\int_t^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt \\
&= \int_0^{\frac{1}{R}} t^{p(\alpha-1)}\left|\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt + \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\int_t^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt \\
&= \int_0^{\frac{1}{R}} t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Big|_{s=\frac{1}{R}}^{s=R}\right|^p dt \\
&\quad + \int_{\frac{1}{R}}^R t^{p(\alpha-1)}\left|\frac{s^{-\alpha-\frac{1}{p}+1}}{-\alpha-\frac{1}{p}+1}\Big|_{s=t}^{s=R}\right|^p dt \\
&= \int_0^{\frac{1}{R}} \frac{1}{|-\alpha-\frac{1}{p}+1|^p} t^{p(\alpha-1)}|R^{-\alpha-\frac{1}{p}+1} - R^{\alpha+\frac{1}{p}-1}|^p dt \\
&\quad + \int_{\frac{1}{R}}^R \frac{1}{|-\alpha-\frac{1}{p}+1|^p} t^{p(\alpha-1)}|R^{-\alpha-\frac{1}{p}+1} - t^{-\alpha-\frac{1}{p}+1}|^p dt \\
&= \frac{1}{|-\alpha-\frac{1}{p}+1|^p}|R^{-\alpha-\frac{1}{p}+1} - R^{\alpha+\frac{1}{p}-1}|^p \int_0^{\frac{1}{R}} t^{p(\alpha-1)} dt \\
&\quad + \int_{\frac{1}{R}}^R \frac{1}{|-\alpha-\frac{1}{p}+1|^p} t^{p(\alpha-1)}|R^{-\alpha-\frac{1}{p}+1} - t^{-\alpha-\frac{1}{p}+1}|^p dt \\
&= +\infty,
\end{aligned}$$

because $p(\alpha-1) < -1$.
If $\alpha + \frac{1}{p} = 1$, then

$$\begin{aligned}
\|Lu_R\|^p_{L^{p,\alpha}(0,+\infty)} &= \int_0^\infty t^{\alpha p}|Lu_R(t)|^p dt = \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_t^\infty u_R(s)ds\right|^p dt \\
&= \int_0^\infty t^{\alpha p}\left|\frac{1}{t}\int_t^\infty s^{-\alpha-\frac{1}{p}}\chi_{(\frac{1}{R},R)}(s)ds\right|^p dt \\
&= \int_0^{\frac{1}{R}} t^{\alpha p}\left|\frac{1}{t}\int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt + \int_{\frac{1}{R}}^R t^{\alpha p}\left|\frac{1}{t}\int_t^R s^{-\alpha-\frac{1}{p}}ds\right|^p dt
\end{aligned}$$

$$= \int_0^{\frac{1}{R}} t^{p(\alpha-1)} \left| \int_{\frac{1}{R}}^R s^{-\alpha-\frac{1}{p}} ds \right|^p dt + \int_{\frac{1}{R}}^R t^{p(\alpha-1)} \left| \int_t^R s^{-\alpha-\frac{1}{p}} ds \right|^p dt$$
$$= \int_0^{\frac{1}{R}} t^{-1} \left| \int_{\frac{1}{R}}^R s^{-1} ds \right|^p dt + \int_{\frac{1}{R}}^R t^{-1} \left| \int_t^R s^{-1} ds \right|^p dt$$
$$= (2\ln R)^p \int_0^{\frac{1}{R}} t^{-1} dt + \int_{\frac{1}{R}}^R t^{-1} (\ln R - \ln t)^p dt = +\infty.$$

Thus, inequality (21) fails for $u_R$.

2. **Case** $\mathbf{p} = +\infty$.
(i)

- We assume that $\alpha < 1$, and we want to show that the operator $H$ is continuous in $L^{\infty,\alpha}(0,+\infty)$, that is

$$\left\| \frac{1}{t} \int_0^t u(s) ds \right\|_{L^{\infty,\alpha}(0,+\infty)} \leq C \|u\|_{L^{\infty,\alpha}(0,+\infty)}. \tag{22}$$

We observe that for all $t > 0$

$$\left| t^\alpha \left( \frac{1}{t} \int_0^t u(s) ds \right) \right| \leq t^{\alpha-1} \int_0^t |u(s)| \frac{s^\alpha}{s^\alpha} ds$$
$$\leq \operatorname*{ess\,sup}_{s>0} |s^\alpha u(s)| t^{\alpha-1} \int_0^t \frac{1}{s^\alpha} ds = \frac{1}{1-\alpha} \|s^\alpha u(s)\|_{L^\infty(0,+\infty)}.$$

Thus, we have

$$\operatorname*{ess\,sup}_{t>0} \left| t^\alpha \left( \frac{1}{t} \int_0^t u(s) ds \right) \right| \leq \frac{1}{1-\alpha} \|u(s)\|_{L^{\infty,\alpha}(0,+\infty)}$$

and hence we get (22).

- We would like to show that if $H$ is continuous in $L^{\infty,\alpha}(0,+\infty)$, then $\alpha < 1$. We prove it by contradiction. Suppose $\alpha \geq 1$. We will show that there exists $u \in L^{\infty,\alpha}(0,+\infty)$ such that the inequality (22) fails.
We take $u(t) = t^{-\alpha}$. Then,

$$\|u\|_{L^{\infty,\alpha}(0,+\infty)} = \operatorname*{ess\,sup}_{t>0} |t^\alpha u(t)| = \operatorname*{ess\,sup}_{t>0} 1 = 1$$

and for $\alpha \geq 1$ we have

$$\|Hu(t)\|_{L^{\infty,\alpha}(0,+\infty)} = \operatorname*{ess\,sup}_{t>0} \left| t^\alpha \left( \frac{1}{t} \int_0^t u(s) ds \right) \right|$$
$$= \operatorname*{ess\,sup}_{t>0} \left| t^{\alpha-1} \int_0^t s^{-\alpha} ds \right| = +\infty.$$

(ii) We assume that $\alpha > 1$, and we want to show that the operator $L$ is continuous in $L^{\infty,\alpha}(0,+\infty)$, that is

$$\left\| \frac{1}{t} \int_t^\infty u(s)ds \right\|_{L^{\infty,\alpha}(0,+\infty)} \leq C\|u\|_{L^{\infty,\alpha}(0,+\infty)}. \tag{23}$$

We observe that for all $t > 0$

$$\begin{aligned}\left| t^\alpha \left( \frac{1}{t} \int_t^\infty u(s)ds \right) \right| &\leq t^{\alpha-1} \int_t^\infty |u(s)| \frac{s^\alpha}{s^\alpha} ds \\ &\leq \operatorname*{ess\,sup}_{s>0} |s^\alpha u(s)| t^{\alpha-1} \int_t^\infty \frac{1}{s^\alpha} ds = \frac{1}{\alpha-1} \|s^\alpha u(s)\|_{L^\infty(0,+\infty)}.\end{aligned}$$

Thus, we have

$$\operatorname*{ess\,sup}_{t>0} \left| t^\alpha \left( \frac{1}{t} \int_t^\infty u(s)ds \right) \right| \leq \frac{1}{1-\alpha} \|u(s)\|_{L^{\infty,\alpha}(0,+\infty)}$$

and hence we get (23).

- We would like to show that if $L$ is continuous in $L^{\infty,\alpha}(0,+\infty)$, then $\alpha > 1$. We prove it by contradiction. We assume that $\alpha \leq 1$. We will show that there exists $u \in L^{\infty,\alpha}(0,+\infty)$ such that the following inequality (23) fails.
We take $u(t) = t^{-\alpha}$. Then,

$$\|u\|_{L^{\infty,\alpha}(0,+\infty)} = \operatorname*{ess\,sup}_{t>0} |t^\alpha u(t)| = \operatorname*{ess\,sup}_{t>0} 1 = 1$$

and for $\alpha \leq 1$ we have

$$\begin{aligned}\|Lu(t)\|_{L^{\infty,\alpha}(0,+\infty)} &= \operatorname*{ess\,sup}_{t>0} \left| t^\alpha \left( \frac{1}{t} \int_t^\infty u(s)ds \right) \right| \\ &= \operatorname*{ess\,sup}_{t>0} \left| t^{\alpha-1} \int_t^\infty s^{-\alpha} ds \right| = +\infty.\end{aligned}$$

□

## 5.2. IMBEDDING RESULTS

**Definition 18.** *Let $\Omega \subseteq \mathbb{R}^n$. We denote by $\mathcal{D}(\Omega)$ the space of all $C^\infty$ functions with compact support in $\Omega$.*

**Remark 4** (Chapter 1, Section 1.3.2 in [3])**.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$. In general, $\mathcal{D}(\Omega)$ is not dense in $W^{s,p}(\Omega)$.*

**Definition 19** (Def. 1.3.2.2 in [3]). *Let $\Omega$ be an open subset of $\mathbb{R}^n$. For $s>0$, we denote by $W_0^{s,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{s,p}(\Omega)$.*

**Theorem 20** (Theorem 3.40 in [12] ). *For $0<\alpha\le\frac{1}{2}$ there holds*

$$H_0^\alpha(0,+\infty)=H^\alpha(0,+\infty).$$

*Proof.* We know that $H_0^\alpha(0,+\infty)$ is the closure of $\mathcal{D}((0,+\infty))$ in $H^\alpha(0,+\infty)$. Therefore, it is enough to show that $\mathcal{D}((0,+\infty))$ is a dense subset of $H^\alpha(0,+\infty)$ in order to get the desired result. We will use the following characterization of the dense subspace of Banach space $X$:

A subset $W\subseteq X$ is dense if and only if $\{g\in X^*:\langle g,u\rangle=0 \text{ for all } u\in W\}=\{0\}$.

Let $\tilde F: H^\alpha(0,+\infty)\to\mathbb{C}$ be a linear, continuous functional and $\tilde F(\phi)=0$ for all $\phi\in\mathcal{D}((0,+\infty))$. We would like to show that $\tilde F\equiv 0$.
First, we note that

$$\tilde F(\chi)=0 \quad \text{if} \quad \chi\in\mathcal{D}([0,+\infty)) \text{ and } \chi(0)=0. \tag{24}$$

Indeed, if $\chi\in\mathcal{D}([0,+\infty))$ and $\chi(0)=0$, then we define

$$\chi_\varepsilon(x)=\begin{cases}0 & \text{if } x\in[0,\varepsilon],\\ \chi(x-\varepsilon) & \text{if } x>\varepsilon.\end{cases}$$

Then $\chi_\varepsilon$ is continuous on $[0,+\infty)$. Moreover, $\chi_\varepsilon\to\chi$ in $H^1(0,+\infty)$, because

$$\begin{aligned}\int_0^\infty|\chi_\varepsilon'(x)-\chi'(x)|^2dx&=\int_0^\varepsilon|\chi'(x)|^2dx+\int_\varepsilon^\infty|\chi'(x-\varepsilon)-\chi'(x)|^2dx\\&=\int_0^\varepsilon|\chi'(x)|^2dx+\int_0^\infty|\chi'(x)-\chi'(x+\varepsilon)|^2dx\to 0 \text{ as } \varepsilon\to 0^+.\end{aligned}$$

Next, if $(\chi_\varepsilon)_{\frac{\varepsilon}{2}}=\eta_{\frac{\varepsilon}{2}}*\chi_\varepsilon$ is a standard mollifier, then $(\chi_\varepsilon)_{\frac{\varepsilon}{2}}\to\chi$ in $H^1(0,+\infty)$ and $(\chi_\varepsilon)_{\frac{\varepsilon}{2}}\in\mathcal{D}((0,+\infty))$. In particular, $(\chi_\varepsilon)_{\frac{\varepsilon}{2}}\to\chi$ in $H^\alpha(0,+\infty)$, because from Corollary 2.8. in [10] we get

$$\|\chi_\varepsilon-\chi\|_{H^\alpha(0,+\infty)}\le\|\chi_\varepsilon-\chi\|_{L^2(0,+\infty)}^{1-\alpha}\|\chi_\varepsilon-\chi\|_{H^1(0,+\infty)}^{\alpha}.$$

From the observation that $(\chi_\varepsilon)_{\frac{\varepsilon}{2}}\to\chi$ in $H^\alpha(0,+\infty)$ and the fact that $\tilde F$ is continuous in $H^\alpha(0,+\infty)$, we get

$$\tilde F(\chi)=\lim_{\varepsilon\to 0}\tilde F((\chi_\varepsilon)_{\frac{\varepsilon}{2}})=0,$$

so we have (24). We recall that for $w\in H^\alpha(\mathbb{R})$ we can write

$$w(x)=w_o(x)+w_e(x) \quad \text{for } x\in\mathbb{R},$$

where

$$w_o(x)=\frac{1}{2}[w(x)-w(-x)]$$

is the odd part of the function $w$ and

$$w_e(x) = \frac{1}{2}[w(x) + w(-x)]$$

is the even part of the function $w$. Hence, we define $F : H^{\alpha}(\mathbb{R}) \to \mathbb{C}$ such that

$$Fw := \tilde{F} w_{e|_{(0,+\infty)}}.$$

Then, $w_{e|_{(0,+\infty)}} \in H^{\alpha}(0,+\infty)$ and $F$ is well defined. We notice that $F$ is a continuous linear operator. Moreover, we observe that

$$\text{if } \phi \in \mathcal{D}(\mathbb{R}) \text{ such that } \phi(0) = 0, \text{ then } F\phi = 0. \tag{25}$$

Indeed, $F\phi = \tilde{F}\phi_{e|_{(0,+\infty)}}$ and $\phi_{e|_{(0,+\infty)}} \in \mathcal{D}([0,+\infty))$ such that $\phi_{e|_{(0,+\infty)}}(0) = 0$, because $0 = \phi(0) = \phi_o(0) + \phi_e(0) = 0 + \phi_e(0) = \phi_e(0)$. From (24) we get $\tilde{F}\phi_{e|_{(0,+\infty)}} = 0$, and hence we obtain (25).

Thus, it is enough to show that $F \equiv 0$. To this purpose, let $\eta \in \mathcal{D}(\mathbb{R})$ and $\eta(0) = 1$. Then for all $\psi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle F, \psi \rangle = \langle F, \psi - \eta\psi(0) \rangle + \langle F, \eta\psi(0) \rangle = \langle F, \eta\psi(0) \rangle.$$

Thus, for all $\psi \in \mathcal{D}(\mathbb{R})$ we obtain

$$\langle F, \psi \rangle = \psi(0)\langle F, \eta \rangle.$$

We put $a := \langle F, \eta \rangle$, so for all $\psi \in \mathcal{D}(\mathbb{R})$

$$\langle F, \psi \rangle = a\psi(0).$$

Hence, we can write $F := a\delta$, where $\delta$ denotes Dirac delta function. Next

$$F \in H^{-\alpha}(\mathbb{R}) \Leftrightarrow \hat{F}(\xi)(1 + |\xi|^2)^{-\frac{\alpha}{2}} \in L^2(\mathbb{R}).$$

However, $\hat{F}(\xi) = a$. Thus,

$$F \in H^{-\alpha}(\mathbb{R}) \Leftrightarrow a(1 + |\xi|^2)^{-\frac{\alpha}{2}} \in L^2(\mathbb{R})$$

$$\Leftrightarrow |a|^2 \int_{\mathbb{R}} (1 + |\xi|^2)^{-\alpha} d\xi < +\infty.$$

We observe that

$$\int_{\mathbb{R}} (1 + |\xi|^2)^{-\alpha} d\xi = +\infty$$

for $\alpha \in (0, \frac{1}{2}]$, so we get $a = 0$. Therefore, for all $\psi \in \mathcal{D}(\mathbb{R})$ we have $\langle F, \psi \rangle = 0$, and hence $F \equiv 0$.

From the fact that $F \equiv 0$, we can deduce that $\tilde{F} \equiv 0$. Indeed, let $\tilde{v} \in H^{\alpha}((0,+\infty))$. We set

$$v(x) = \begin{cases} \tilde{v}(x) & \text{if } x > 0, \\ \tilde{v}(-x) & \text{if } x < 0. \end{cases}$$

We show that $v \in H^\alpha(\mathbb{R})$.

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2\alpha}}dxdy &= \int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^{+\infty}\int_{-\infty}^{0} \frac{|\tilde{v}(-x)-\tilde{v}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_{-\infty}^{0}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(-y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{|\tilde{v}(-x)-\tilde{v}(-y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&= \int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x+y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x+y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\leq 4\int_0^{+\infty}\int_0^{+\infty} \frac{|\tilde{v}(x)-\tilde{v}(y)|^2}{|x-y|^{1+2\alpha}}dxdy,
\end{aligned}
$$

where in the last inequality we used the fact that $|x-y| \leq |x+y|$ for all $x, y \geq 0$. Moreover,

$$\|v\|^2_{L^2(\mathbb{R})} = \int_0^{+\infty} |\tilde{v}(x)|^2 dx + \int_{-\infty}^{0} |\tilde{v}(-x)|^2 dx = 2\int_0^{+\infty} |\tilde{v}(x)|^2 dx = 2\|\tilde{v}\|^2_{L^2(\mathbb{R})}.$$

Thus,

$$\|v\|^2_{H^\alpha(\mathbb{R})} \leq 4\|\tilde{v}\|^2_{H^\alpha((0,+\infty))}.$$

Furthermore, from definition of $F$ we obtain

$$\tilde{F}\tilde{v} = Fv = 0,$$

because $F \equiv 0$. To sum up, we have $\tilde{F}\tilde{v} = 0$ for all $\tilde{v} \in H^\alpha((0,+\infty))$. Hence, $\tilde{F} \equiv 0$.

□

**Remark 5.** *Let $T > 0$. Proposition 20 holds also for $\Omega = (-\infty, T)$ or $\Omega = (-\infty, 0)$ instead of $\Omega = (0, +\infty)$. Hence,*

$$H_0^\alpha(0,T) = H^\alpha(0,T) \quad \textit{for} \quad 0 < \alpha \leq \frac{1}{2}.$$

*Indeed, let $f \in H^\alpha(0,T)$ and let $\eta \in \mathcal{D}(\mathbb{R})$ such that* $\operatorname{supp}\eta \subseteq [0, \frac{3}{5}T]$ *and $\eta \equiv 1$ on $[0, \frac{2}{5}T]$. Then, we have*

$$f(t) = f(t)\eta(t) + f(t)(1-\eta)(t).$$

*Moreover, we observe that* $f\eta \in H^{\alpha}(0,+\infty)$ *and* $f(1-\eta) \in H^{\alpha}(-\infty,T)$. *Thus, there exists* $\varphi_1 \in \mathcal{D}((0,+\infty))$ *such that* $\operatorname{supp}\varphi_1 \subseteq (0,\frac{4}{5}T)$ *and* $\|\varphi_1 - f\eta\|_{H^{\frac{1}{2}}(0,+\infty)} < \frac{\varepsilon}{2}$. *Furthermore, there exists* $\varphi_2 \in \mathcal{D}((-\infty,T))$ *such that* $\operatorname{supp}\varphi_2 \subseteq (\frac{1}{5}T,T)$ *and* $\|\varphi_2 - f(1-\eta)\|_{H^{\frac{1}{2}}(-\infty,T)} < \frac{\varepsilon}{2}$. *Hence,* $\varphi_1+\varphi_2 \in \mathcal{D}((0,T))$ *and*

$$\|(\varphi_1+\varphi_2)-f\|_{H^{\alpha}(0,T)} < \varepsilon.$$

It is worth mentioning that the results presented in Theorem 20 and Remark 5 hold for all $1 \le p < +\infty$ and for any bounded and open subset of $\mathbb{R}^n$, which we present in the following theorem:

**Theorem 21** (Thr. 1.4.2.4 in [3] )**.** *Let* $\Omega$ *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz boundary; then* $\mathcal{D}(\Omega)$ *is dense in* $W^{s,p}(\Omega)$ *for* $0 < s \le \frac{1}{p}$.

**Corollary 22** (Chapter 1, Section 1.4.2 in [3])**.** *Under the assumptions of Theorem 21,* $W_0^{s,p}(\Omega)$ *is the same space as* $W^{s,p}(\Omega)$ *when* $0 < s \le \frac{1}{p}$.

**Theorem 23** (Thr. 1.4.4.4 in [3])**.** *Let* $1 \le p < +\infty$. *We assume that* $s \ge 0$ *and* $s = m+\sigma$, *where* $m$ *is an integer and* $\sigma \in [0,1)$. *For all* $u \in W_0^{s,p}(0,+\infty)$ *such that* $s-\frac{1}{p}$ *is not an integer, the following property holds:*

$$x^{-s+\alpha}u^{(\alpha)} \in L^p(0,+\infty),$$

*and we have the estimate*

$$\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} \le C(s,p,\sigma)\|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)} \tag{26}$$

*for all* $\alpha \in \mathbb{N}$ *such that* $\alpha \le s$. *In particular, for all* $\alpha \in \mathbb{N}$ *such that* $\alpha \le s$ *the following estimate holds:*

$$\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} \le C(s,p,\sigma)\|u\|_{W^{s,p}(0,+\infty)}. \tag{27}$$

*Proof.* Suppose that we have

$$\|x^{-\sigma}v\|_{L^p(0,+\infty)} \le C(s,p,\sigma)\|v\|_{W^{\sigma,p}(0,+\infty)} \quad \text{for} \quad v \in \mathcal{D}((0,+\infty)). \tag{28}$$

If $u \in \mathcal{D}((0,+\infty))$, then (28) with $v = u^{(m)}$ gives (26) for $u \in \mathcal{D}((0,+\infty))$ and $\alpha = m$. Furthermore, using Hardy's inequality we will get (26) for all $\alpha \in \mathbb{N}$ such that $\alpha < m$. To this purpose, we observe that for all $\alpha \in \mathbb{N}$ such that $\alpha < m$ we have $-s+\alpha+1+\frac{1}{p} < 1$, because

$$-s+\alpha+1+\frac{1}{p} < 1 \Leftrightarrow s-\alpha > \frac{1}{p} \Leftrightarrow (m-\alpha)+\sigma > \frac{1}{p}.$$

The last inequality holds, because if $p > 1$, then $\sigma \in [0,1)$ and $(m-\alpha)+\sigma \ge 1+\sigma \ge 1 > \frac{1}{p}$. If $p = 1$, then $s$ can not be an integer, so in this case $\sigma \in (0,1)$ and $(m-\alpha)+\sigma \ge 1+\sigma > 1 = \frac{1}{p}$.

Due to the above observation, we can use Hardy's inequality with the operator $H$. Thus, for all $\alpha \in \mathbb{N}$ such that $\alpha < m$ and $u \in \mathcal{D}((0,+\infty))$ we have

$$\begin{aligned}
\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} &= \left\| x^{-s+\alpha+1}\left(\frac{1}{x}\int_0^x u^{(\alpha+1)}(s)ds\right)\right\|_{L^p(0,+\infty)} \\
&= \|x^{-s+\alpha+1}H(u^{(\alpha+1)})(x)\|_{L^p(0,+\infty)} \\
&\leq \frac{1}{|-s+\alpha+\frac{1}{p}|}\|x^{-s+\alpha+1}u^{(\alpha+1)}\|_{L^p(0,+\infty)} \\
&\leq \ldots \leq \prod_{\beta=\alpha}^{m-1}\frac{1}{|-s+\beta+\frac{1}{p}|}\|x^{-s+m}u^{(m)}\|_{L^p(0,+\infty)} \\
&= C(s,p)\|x^{-\sigma}u^{(m)}\|_{L^p(0,+\infty)}.
\end{aligned}$$

Hence, we get

$$\|x^{-s+\alpha}u^{(\alpha)}u\|_{L^p(0,+\infty)} \leq C(s,p)\|x^{-\sigma}u^{(m)}\|_{L^p(0,+\infty)} \tag{29}$$

for all $\alpha \in \mathbb{N}$ such that $\alpha < m$ and $u \in \mathcal{D}((0,+\infty))$. Combining (28) with (29), we get

$$\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} \leq C(s,p,\sigma)\|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)} \quad \text{for } u \in \mathcal{D}((0,+\infty)) \tag{30}$$

and for all $\alpha \in \mathbb{N}$ such that $\alpha \leq m$.

Now, we are going to show (28).

**Remark**: If $s$ is an integer, then $\sigma = 0$ and $\|x^{-\sigma}v\|_{L^p(0,+\infty)} = \|v\|_{L^p(0,+\infty)}$. Thus, if $s$ is an integer, we immediately have (28) with $C(s,p,\sigma) \equiv 1$.

Hence, we have to derive (28) only in the case when $s = m+\sigma$ and $s$ is not an integer.

1. **Case**: $\sigma < \mathbf{\frac{1}{p}}$. Let

$$w(x) = \frac{1}{x}\int_0^x [v(t)-v(x)]dt. \tag{31}$$

We have the following identity:

$$v(x) = -w(x) + \int_x^\infty \frac{w(y)}{y}dy, \tag{32}$$

which we will show below. Indeed,

$$\begin{aligned}
-w(x)+\int_x^\infty \frac{w(y)}{y}dy &= -\frac{1}{x}\int_0^x [v(t)-v(x)]dt + \int_x^\infty \frac{\frac{1}{y}\int_0^y [v(t)-v(y)]dt}{y}dy \\
&= -\frac{1}{x}\int_0^x v(t)dt + v(x) + \int_x^\infty \frac{1}{y^2}\int_0^y v(t)dtdy - \int_x^\infty \frac{v(y)}{y}dy \\
&= -\frac{1}{x}\int_0^x v(t)dt + v(x) + \int_x^\infty \frac{1}{y^2}\int_0^x v(t)dtdy \\
&\quad + \int_x^\infty \frac{1}{y^2}\int_x^y v(t)dtdy - \int_x^\infty \frac{v(y)}{y}dy \\
&= -\frac{1}{x}\int_0^x v(t)dt + v(x) + \int_0^x v(t)dt\int_x^\infty \frac{1}{y^2}dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_x^{\infty}\int_t^{\infty}\frac{1}{y^2}dyv(t)dt-\int_x^{\infty}\frac{v(y)}{y}dy\\
&=-\frac{1}{x}\int_0^x v(t)dt+v(x)+\frac{1}{x}\int_0^x v(t)dt+\int_x^{\infty}\frac{v(t)}{t}dt-\int_x^{\infty}\frac{v(y)}{y}dy\\
&=v(x).
\end{aligned}
$$

Hence, we have (32). We want to show (28). First, we show the following estimate:

$$
\|x^{-\sigma}w\|_{L^p(0,+\infty)}\leq\|v\|_{W^{\sigma,p}(0,+\infty)}, \tag{33}
$$

To this purpose, we perform the following calculations:

$$
\begin{aligned}
\int_0^{\infty}|x^{-\sigma}w(x)|^p dx&=\int_0^{\infty}x^{-\sigma p}\left|\frac{1}{x}\int_0^x[v(t)-v(x)]dt\right|^p dx\\
&\leq\int_0^{\infty}x^{-\sigma p-1}\int_0^x|v(t)-v(x)|^p dtdx=\int_0^{\infty}\int_0^x\frac{|v(t)-v(x)|^p}{x^{\sigma p+1}}dtdx\\
&\leq\int_0^{\infty}\int_0^x\frac{|v(t)-v(x)|^p}{(x-t)^{\sigma p+1}}dtdx=\int_0^{\infty}\int_0^x\frac{|v(t)-v(x)|^p}{|x-t|^{\sigma p+1}}dtdx\\
&\leq\int_0^{\infty}\int_0^{\infty}\frac{|v(t)-v(x)|^p}{|x-t|^{\sigma p+1}}dtdx\leq\|v\|^p_{W^{\sigma,p}(0,+\infty)},
\end{aligned}
$$

where we used Jensen's inequality. Thus, we have (33). We notice that if $\sigma<\frac{1}{p}$, then $(-\sigma+1)+\frac{1}{p}>1$. Hence, from Theorem 17 we deduce that operator $L$ is continuous in $L^{p,1-\sigma}((0,+\infty))$ in the case when $\sigma<\frac{1}{p}$. So, using Hardy's inequality we get

$$
\left\|x^{-\sigma}\int_x^{\infty}\frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)}\leq\frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)}. \tag{34}
$$

Indeed,

$$
\begin{aligned}
\left\|x^{-\sigma}\int_x^{\infty}\frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)}&=\left\|x^{-\sigma+1}\left(\frac{1}{x}\int_x^{\infty}\frac{w(y)}{y}dy\right)\right\|_{L^p(0,+\infty)}\\
&=\left\|x^{-\sigma+1}L\Big(\frac{w(y)}{y}\Big)(x)\right\|_{L^p(0,+\infty)}\\
&\leq\frac{p}{1-p\sigma}\left\|x^{-\sigma+1}\frac{w(x)}{x}\right\|_{L^p(0,+\infty)}\\
&=\frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)}.
\end{aligned}
$$

Due to (32), (33) and (34) we have

$$
\|x^{-\sigma}v\|_{L^p(0,+\infty)}\leq\|x^{-\sigma}w(x)\|_{L^p(0,+\infty)}+\left\|x^{-\sigma}\int_x^{\infty}\frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)}
$$

$$\leq \|v\|_{W^{\sigma,p}(0,+\infty)} + \frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)}$$
$$\leq \left(1+\frac{p}{1-p\sigma}\right)\|v\|_{W^{\sigma,p}(0,+\infty)},$$

provided that $\sigma < \frac{1}{p}$. Thus, we get (28) for $\sigma < \frac{1}{p}$.

2. **Case**: $\sigma > \frac{1}{\mathbf{p}}$. Notice that identity (32) is inconclusive if $\sigma > \frac{1}{p}$. Our aim is to show (28) by using Hardy's inequality. For $\sigma > \frac{1}{p}$ Theorem 17 states that operator $H$ is continuous in $L^{p,1-\sigma}((0,+\infty))$. Thus, we have to derive the identity similar to (32), but in this identity the element $H\left(\frac{w(y)}{y}\right)(x)$ must appear. Notice that

$$\begin{aligned}\int_0^\infty \frac{w(y)}{y}dy &= \int_0^\infty \frac{\frac{1}{y}\int_0^y [v(t)-v(y)]dt}{y}dy\\ &= \int_0^\infty \frac{1}{y^2}\int_0^y v(t)dtdy - \int_0^\infty \frac{v(y)}{y}dy\\ &= \int_0^\infty v(t)\int_t^\infty \frac{1}{y^2}dydt - \int_0^\infty \frac{v(y)}{y}dy = 0.\end{aligned}$$

Hence, from the above observation and the identity (32) we get

$$v(x) = -w(x) - \int_0^x \frac{w(y)}{y}dy. \tag{35}$$

From Case 1. we have (33). Thus, it is enough to show the following estimate:

$$\left\|x^{-\sigma}\int_0^x \frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)} \leq \frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)} \tag{36}$$

in order to get (28). Indeed, using Hardy's inequality we get

$$\begin{aligned}\left\|x^{-\sigma}\int_0^x \frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)} &= \left\|x^{-\sigma+1}\left(\frac{1}{x}\int_0^x \frac{w(y)}{y}dy\right)\right\|_{L^p(0,+\infty)}\\ &= \left\|x^{-\sigma+1}H\left(\frac{w(y)}{y}\right)(x)\right\|_{L^p(0,+\infty)}\\ &\leq \frac{p}{1-p\sigma}\left\|x^{-\sigma+1}\frac{w(x)}{x}\right\|_{L^p(0,+\infty)}\\ &= \frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)}.\end{aligned}$$

Thus, again from (33), (35) and (36) we get

$$\begin{aligned}\|x^{-\sigma}v\|_{L^p(0,+\infty)} &\leq \|x^{-\sigma}w(x)\|_{L^p(0,+\infty)} + \left\|x^{-\sigma}\int_0^x \frac{w(y)}{y}dy\right\|_{L^p(0,+\infty)}\\ &\leq \|v\|_{W^{\sigma,p}(0,+\infty)} + \frac{p}{1-p\sigma}\|x^{-\sigma}w\|_{L^p(0,+\infty)} \leq \left(1+\frac{p}{1-p\sigma}\right)\|v\|_{W^{\sigma,p}(0,+\infty)}.\end{aligned}$$

This gives us (28) for $\sigma > \frac{1}{p}$.
To sum up, from Case 1. and Case 2. we get (28) for $\sigma \neq \frac{1}{p}$. Further, we recall that (28) implies (30), that is

$$\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} \leq C(s,p,\sigma)\|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)}$$

for $u \in \mathcal{D}((0,+\infty))$ and for all $\alpha \in \mathbb{N}$ such that $\alpha \leq s$.
Let $u \in W_0^{m+\sigma,p}(0,+\infty)$. Since $\mathcal{D}((0,+\infty))$ is dense in $W_0^{m+\sigma,p}(0,+\infty)$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}((0,+\infty))$ such that $\|u_n - u\|_{W^{m+\sigma,p}(0,+\infty)} \to 0$ when $n \to \infty$. We can rewrite (30) replacing $u$ with $u_l - u_k$

$$\|x^{-s+\alpha}(u_l^{(\alpha)} - u_k^{(\alpha)})\|_{L^p(0,+\infty)} \leq C(s,p,\sigma)\|u_l^{(m)} - u_k^{(m)}\|_{W^{\sigma,p}(0,+\infty)}.$$

Due to the fact that $\|u_n - u\|_{W^{m+\sigma,p}(0,+\infty)} \to 0$ when $n \to \infty$, we have $\|u_n^{(m)} - u^{(m)}\|_{W^{\sigma,p}(0,+\infty)} \to 0$ when $n \to \infty$. Hence, $\{u_n^{(m)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_0^{\sigma,p}(0,+\infty)$. From the above inequality we can deduce that $\{x^{-s+\alpha}u_n^{(\alpha)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(0,+\infty)$, so it is convergent in $L^p(0,+\infty)$ to some $v \in L^p(0,+\infty)$. On the other hand, $u_n^{(\alpha)} \to u^{(\alpha)}$ in $L^p(0,+\infty)$. Hence, $v = x^{-s+\alpha}u^{(\alpha)}$.
Now, we observe that for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$

$$\|x^{-s+\alpha}u_n^{(\alpha)} - x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} < \varepsilon \tag{37}$$

and there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$

$$\|u_n^{(m)} - u^{(m)}\|_{W^{\sigma,p}(0,+\infty)} < \varepsilon. \tag{38}$$

Moreover, the inequality (30) holds for terms of the sequence $\{u_n\}_{n\in\mathbb{N}}$

$$\|x^{-s+\alpha}u_n^{(\alpha)}\|_{L^p(0,+\infty)} \leq C(s,p,\sigma)\|u_n^{(m)}\|_{W^{\sigma,p}(0,+\infty)}. \tag{39}$$

We set $N_3 = \max\{N_1, N_2\}$ and we choose some $n > N_3$. Thus, from (37), (38) and (39) we get

$$\begin{aligned}
\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} &\leq \|x^{-s+\alpha}u^{(\alpha)} - x^{-s+\alpha}u_n^{(\alpha)}\|_{L^p(0,+\infty)} + \|x^{-s+\alpha}u_n^{(\alpha)}\|_{L^p(0,+\infty)} \\
&\leq \varepsilon + C(s,p,\sigma)\|u_n^{(m)}\|_{W^{\sigma,p}(0,+\infty)} \\
&\leq \varepsilon + C(s,p,\sigma)\Big(\|u_n^{(m)} - u^{(m)}\|_{W^{\sigma,p}(0,+\infty)} + \|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)}\Big) \\
&\leq \varepsilon + C(s,p,\sigma)\Big(\varepsilon + \|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)}\Big) \\
&= \varepsilon(C(s,p,\sigma)+1) + C(s,p,\sigma)\|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)}
\end{aligned}$$

for all $\varepsilon > 0$. Going to the limit with $\varepsilon \to 0$, we obtain

$$\|x^{-s+\alpha}u^{(\alpha)}\|_{L^p(0,+\infty)} \leq C(s,p,\sigma)\|u^{(m)}\|_{W^{\sigma,p}(0,+\infty)}$$

for all $u \in W_0^{m+\sigma,p}(0,+\infty)$. This gives us (26) for all $u \in W_0^{m+\sigma,p}(0,+\infty)$, provided that $s - \frac{1}{p}$ is not an integer. Furthermore, inequality (27) is a consequence of the definition of the norm $\|\cdot\|_{W^{s,p}(0,+\infty)}$. $\square$

**Lemma 24.** *Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary $\Gamma$. Then for all $u \in W_0^{s,p}(\Omega)$ such that $s - \frac{1}{p}$ is not an integer the following property holds:*

$$dist(y,\Gamma)^{-s+|\alpha|} D^\alpha u \in L^p(\Omega) \tag{40}$$

*for all $|\alpha| \le s$, and we have the following estimate:*

$$\|dist(y,\Gamma)^{-s+|\alpha|} D^\alpha u\|_{L^p(\Omega)} \le C(s,p,\sigma)\|u\|_{W^{s,p}(\Omega)}. \tag{41}$$

*Proof.* Let $u \in \mathcal{D}(\Omega)$. We set $x \in \Gamma$ and let $V$ be an open neighbourhood of $x$ in $\mathbb{R}^n$. Without loss of generality, we may assume that

1. $V$ has a form of a hybercube in some local coordinates $\{y_1, \dots, y_n\}$

$$V = \{(y_1, \dots, y_n) : -a_j < y_j < a_j, 1 \le j \le n\};$$

2. there exists a Lipschitz function $\varphi$ defined in

$$V' = \{(y_1, \dots, y_{n-1}) : -a_j < y_j < a_j, 1 \le j \le n-1\}$$

and such that

$$|\varphi(y')| \le \frac{a_n}{2} \quad \text{for every } y' = (y_1, \dots, y_{n-1}) \in V',$$
$$\Omega \cap V = \{y = (y', y_n) \in V : y_n < \varphi(y')\},$$
$$\Gamma \cap V = \{y = (y', y_n) \in V : y_n = \varphi(y')\}.$$

Using a partition of unity, we may assume that supp $u \subseteq V$. For $y' \in V'$ we set

$$u_{y'}(t) = u(y', \varphi(y') - t).$$

We notice that for almost all $y' \in V'$ we have $u_{y'} \in W_0^{s,p}(\mathbb{R}_+)$. Indeed, we have

$$\begin{aligned}
\|u_{y'}\|^p_{W^{s,p}(0,+\infty)} &= \|u_{y'}\|^p_{W^{m,p}(0,+\infty)} + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u_{y'}^{(m)}(t_1) - u_{y'}^{(m)}(t_2)|^p}{|t_1 - t_2|^{n+\sigma p}} dt_1 dt_2 \\
&= \sum_{\alpha \le m} \int_{\mathbb{R}_+} |u_{y'}^{(\alpha)}(t)|^p dt + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|u_{y'}^{(m)}(t_1) - u_{y'}^{(m)}(t_2)|^p}{|t_1 - t_2|^{n+\sigma p}} dt_1 dt_2 \\
&= \sum_{\alpha \le m} \int_{\mathbb{R}_+} |\frac{\partial^\alpha}{\partial y_n} u(y', \varphi(y') - t)|^p dt \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{|\frac{\partial^m}{\partial y_n} u(y', \varphi(y') - t_1) - \frac{\partial^m}{\partial y_n} u(y', \varphi(y') - t_2)|^p}{|t_1 - t_2|^{n+\sigma p}} dt_1 dt_2 \\
&= \sum_{\alpha \le m} \int_{y_n < \varphi(y')} |\frac{\partial^\alpha}{\partial y_n} u(y', y_n)|^p dy_n
\end{aligned}$$

$$+\int_{y_n<\varphi(y')}\int_{z_n<\varphi(y')}\frac{|\frac{\partial^m}{\partial y_n}u(y',y_n)-\frac{\partial^m}{\partial z_n}u(y',z_n)|^p}{|y_n-z_n|^{n+\sigma p}}dy_ndz_n<+\infty$$

for almost every $y'\in V'$, because

$$\|u\|^p_{W^s_p(\Omega)}=\sum_{|\beta|\le m}\int_{\Omega\cap V}|D^\beta u(y)|^pdy+\sum_{|\beta|=m}\int_{\Omega\cap V}\int_{\Omega\cap V}\frac{|D^\beta u(y)-D^\beta u(z)|^p}{|y-z|^{n+\sigma p}}dydz<+\infty.$$

From Theorem 23 we get $t^{-s}u_{y'}\in L^p(\mathbb{R}_+)$ with

$$\|t^{-s}u_{y'}\|^p_{L^p(\mathbb{R}_+)}\le K^p\|u_{y'}\|^p_{W^s_p(\mathbb{R}_+)},$$

where $K$ does not depend on $y'$. We integrate this inequality in $y'$. Then, using the substitution $y_n=\varphi(y')-t$, we have

$$\begin{aligned}\int_{V'}\int_{R_+}t^{-sp}|u_{y'}|^pdtdy'&=\int_{V'}\int_{R_+}t^{-sp}|u(y',\varphi(y')-t)|^pdtdy'\\&=C_1\int_{V'}\int_{y_n<\varphi(y')}[\varphi(y')-y_n]^{-sp}|u(y',y_n)|^pdy_ndy'\\&=C_1\int_{\Omega\cap V}(\varphi(y')-y_n)^{-sp}|u(y)|^pdy\end{aligned}$$

and

$$\begin{aligned}\int_{V'}\|u_{y'}\|^p_{W^{s,p}(0,+\infty)}dy'&=\int_{V'}\|u_{y'}\|^p_{W^{m,p}(0,+\infty)}dy'+\int_{V'}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|u^{(m)}_{y'}(t_1)-u^{(m)}_{y'}(t_2)|^p}{|t_1-t_2|^{n+\sigma p}}dt_1dt_2dy'\\&=\int_{V'}\sum_{\alpha\le m}\int_{\mathbb{R}_+}|u^{(\alpha)}_{y'}(t)|^pdtdy'\\&\quad+\int_{V'}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|u^{(m)}_{y'}(t_1)-u^{(m)}_{y'}(t_2)|^p}{|t_1-t_2|^{n+\sigma p}}dt_1dt_2dy'\\&=\int_{V'}\sum_{\alpha\le m}\int_{\mathbb{R}_+}|\frac{\partial^\alpha}{\partial y_n}u(y',\varphi(y')-t)|^pdtdy'\\&\quad+\int_{V'}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\frac{|\frac{\partial^m}{\partial y_n}u(y',\varphi(y')-t_1)-\frac{\partial^m}{\partial y_n}u(y',\varphi(y')-t_2)|^p}{|t_1-t_2|^{n+\sigma p}}dt_1dt_2dy'\\&=C_2\int_{V'}\sum_{\alpha\le m}\int_{y_n<\varphi(y')}|\frac{\partial^\alpha}{\partial y_n}u(y',y_n)|^pdy_ndy'\\&\quad+C_3\int_{V'}\int_{y_n<\varphi(y')}\int_{z_n<\varphi(y')}\frac{|\frac{\partial^m}{\partial y_n}u(y',y_n)-\frac{\partial^m}{\partial z_n}u(y',z_n)|^p}{|y_n-z_n|^{n+\sigma p}}dy_ndz_ndy'\\&\le C_2\sum_{|\beta|\le m}\int_{\Omega\cap V}|D^\beta u(y)|^pdy\\&\quad+C_3\sum_{|\beta|=m}\int_{\Omega\cap V}\int_{\Omega\cap V}\frac{|D^\beta u(y)-D^\beta u(z)|^p}{|y-z|^{n+\sigma p}}dydz\end{aligned}$$

$$\leq \max\{C_2, C_3\}\|u\|^p_{W^s_p(\Omega)},$$

where $C_1, C_2$ and $C_3$ depend only on the boundary of $\Omega$. Therefore, we get

$$\|(\varphi(y') - y_n)^{-s} u\|_{L^p(\Omega)} \leq C\|u\|_{W^s_p(\Omega)},$$

where $C$ depends on the boundary of $\Omega$. Furthermore, for $x \in \Gamma$ we have $\varphi(x') = x_n$, and using the fact that $\varphi$ is a Lipschitz function with constant $M$, we obtain

$$\begin{aligned}|\varphi(y') - y_n| &= |x_n - y_n + \varphi(y') - \varphi(x')| \leq |x_n - y_n| + |\varphi(y') - \varphi(x')| \\ &\leq |x_n - y_n| + M|x' - y'| \leq (M+1)|x - y| \leq \sqrt{2(1+M^2)}|x - y|.\end{aligned}$$

Thus, we obtain

$$\operatorname{dist}(y, \Gamma) = \inf_{x \in \Gamma} |x - y| \geq \frac{|\varphi(y') - y_n|}{\sqrt{2(1+M^2)}}$$

for $y \in \Omega$. Due to the above observation, we obtain

$$\|\operatorname{dist}(y, \Gamma)^{-s} u\|_{L^p(\Omega)} \leq C\|u\|_{W^s_p(\Omega)}.$$

Thus, we have (40) for $|\alpha| = 0$. Performing similar calculations for $D^\alpha u$, when $|\alpha| \leq s$, we get the desired result.

□

# 6. PROOF OF THE MAIN RESULT

Firstly, we need to show the following proposition, which then we use in the proof of Theorem 6:

**Proposition 25.** *Let $T > 0$. We denote*

$$\widetilde{H}^1(-T,T) = \{f \in H^1(-T,T) : f(t) = 0 \quad \text{for} \quad t \in (-T,0)\}$$

*and*

$$\widetilde{L}^2(-T,T) = \{f \in L^2(-T,T) : f(t) = 0 \quad \text{for} \quad t \in (-T,0)\}.$$

*Then, for $\alpha \in (0,1)$, the map*

$$f \mapsto \tilde{f} = \begin{cases} f(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0) \end{cases}$$

*is an isomorphism from $[L^2(0,T),\, {}_0H^1(0,T)]_\alpha$ onto $[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$, i.e.*

$$[L^2(0,T),\, {}_0H^1(0,T)]_\alpha \cong [\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha. \tag{42}$$

*Proof.* 1. In the first step, we show that the map

$$f \mapsto \tilde{f} = \begin{cases} f(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0) \end{cases}$$

defines an isomorphism from ${}_0H^1(0,T)$ onto $\widetilde{H}^1(-T,T)$, i.e.

$${}_0H^1(0,T) \cong \widetilde{H}^1(-T,T).$$

To this purpose, we must show that $f \mapsto \tilde{f}$ is the bijection between ${}_0H^1(0,T)$ and $\widetilde{H}^1(-T,T)$. We take $f \in {}_0H^1(0,T)$. It is easy to show that $\tilde{f} \in H^1(-T,T)$. Indeed, if $f \in {}_0H^1(0,T)$, then $f \in L^2(0,T)$. Thus, $\tilde{f} \in L^2(-T,T)$, because $\|\tilde{f}\|_{L^2(-T,T)} = \|f\|_{L^2(0,T)}$. Moreover, we know that there exists $f'$ in a weak sense and $f' \in L^2(0,T)$ and $f(0) = 0$. We calculate $(\tilde{f})'$ in a weak sense. Let $\varphi \in C_c^\infty((-T,T))$. Then

$$\begin{aligned} \int_{-T}^{T} \tilde{f}(t)\varphi'(t)dt &= \int_0^T f(t)\varphi'(t)dt = f(T)\varphi(T) - f(0)\varphi(0) - \int_0^T f'(t)\varphi(t)dt \\ &= -\int_0^T f'(t)\varphi(t)dt = -\int_{-T}^{T} \widetilde{f'}(t)\varphi(t)dt, \end{aligned}$$

where we used the fact that $f(0) = 0$. Hence, $(\tilde{f})' = \widetilde{f'}$ in a weak sense. Moreover, using the above observation we get

$$\begin{aligned} \|\tilde{f}\|^2_{H^1(-T,T)} &= \int_{-T}^{T} |\tilde{f}(t)|^2 dt + \int_{-T}^{T} |(\tilde{f})'(t)|^2 dt = \int_{-T}^{T} |\tilde{f}(t)|^2 dt + \int_{-T}^{T} |\widetilde{f'}(t)|^2 dt \\ &= \int_0^T |f(t)|^2 dt + \int_0^T |f'(t)|^2 dt = \|f\|^2_{H^1(0,T)} < +\infty. \end{aligned}$$

Therefore, the map $f \mapsto \tilde{f}$ is an isometry between ${}_0H^1(0,T)$ and $\widetilde{H}^1(-T,T)$, so it is injective. Furthermore, it is surjective. Indeed, we take arbitrary $g \in \widetilde{H}^1(-T,T)$. Then we know that $g \in H^1(-T,T)$ and $g(t) = 0$ for $t \in (-T,0)$. Hence, $g_{|_{(0,T)}} \in H^1(0,T)$, because

$$\|g_{|_{(0,T)}}\|_{H^1(0,T)} = \|g\|_{H^1(-T,T)}.$$

Moreover, from Theorem 5. in Chapter 5.6.2 in [2] we know that if $g \in H^1(-T,T)$, then there exists a function $g^* \in C^{0,\frac{1}{2}}([-T,T])$ such that $g = g^*$ almost everywhere. Hence, $g$ has a continuous representative $g^*$, which is defined on $[-T,T]$. Further, $g^*(t) = 0$ for $t \in (-T,0)$. Due to the fact that $g^*$ is continuous on $[-T,T]$, we deduce that $g^*(0) = 0$. Hence, $g(0) = 0$ in a trace sense. Thus, $g_{|_{(0,T)}} \in {}_0H^1(0,T)$ and $\tilde{g}_{|_{(0,T)}} = g$ almost everywhere. Therefore,

$${}_0H^1(0,T) \cong \widetilde{H}^1(-T,T).$$

2. Now, we will show that

$$f \mapsto \tilde{f} = \begin{cases} f(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0) \end{cases}$$

defines an isomorphism from $L^2(0,T)$ onto $\widetilde{L}^2(-T,T)$, i.e.

$$L^2(0,T) \cong \widetilde{L}^2(-T,T).$$

We can see that $f \mapsto \tilde{f}$ is a bijection between $L^2(0,T)$ and $\widetilde{L}^2(-T,T)$. Firstly,

$$\|\tilde{f}\|_{\widetilde{L}^2(-T,T)} = \|f\|_{L^2(0,T)},$$

so map $f \mapsto \tilde{f}$ is an isometry. It is also surjective, beacause for $g \in \widetilde{L}^2(-T,T)$ we can take $g_{|_{(0,T)}} \in L^2(0,T)$ and $\tilde{g}_{|_{(0,T)}} = g$ almost everywhere. Therefore, $f \mapsto \tilde{f}$ is an isometric isomorphism and $L^2(0,T) \cong \widetilde{L}^2(-T,T)$.
3. In the third step, we show that

$$\mathcal{F}(L^2(0,T),\ {}_0H^1(0,T)) \cong \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)), \tag{43}$$

where $\mathcal{F}(X_0,X_1)$ is defined for an arbitrary interpolation pair $(X_0,X_1)$ in Definition 2. We take $g \in \mathcal{F}(L^2(0,T),\ {}_0H^1(0,T))$. From Definition 2, we know that

$$g : S \to\ L^2(0,T) +\ {}_0H^1(0,T)$$

and

1. g is continuous and bounded in $S$,
2. g is analytic in $S^0$,
3. $g(it) \in L^2(0,T)$ and $g(it+1) \in\ {}_0H^1(0,T)$ for all $t \in \mathbb{R}$,
4. functions $t \mapsto g(it)$ and $t \mapsto g(it+1)$ are bounded and continuous with respect to the spaces $L^2(0,T)$ and ${}_0H^1(0,T)$, respectively.

For all $z \in S$, we define

$$\tilde{g}(z) = \widetilde{g(z)},$$

where

$$\widetilde{g(z)}(t) = \begin{cases} g(z)(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0). \end{cases}$$

We want to obtain that

$$g \mapsto \tilde{g}$$

is an isometric isomorphism between $\mathcal{F}(L^2(0,T),\ {}_0H^1(0,T))$ and $\mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T))$. For all $z \in S$, we have $g(z) = u_{1,z} + u_{2,z}$, where $u_{1,z} \in L^2(0,T)$ and $u_{2,z} \in\ {}_0H^1(0,T)$. Hence,

$$\widetilde{g(z)} = \widetilde{u_{1,z}} + \widetilde{u_{2,z}},$$

where $\widetilde{u_{1,z}} \in \widetilde{L}^2(-T,T)$ and $\widetilde{u_{2,z}} \in \widetilde{H}^1(-T,T)$. As a result, we get

$$\widetilde{g}(z) = \widetilde{u_{1,z}} + \widetilde{u_{2,z}}.$$

We want to show that $\tilde{g}$ is continuous in $S$. Let $z_0 \in S$. We have

$$\begin{aligned}
&\|\tilde{g}(z)-\tilde{g}(z_0)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}\\
&=\inf\{\|\widetilde{u_{1,z}}-\widetilde{u_{1,z_0}}\|_{\widetilde{L}^2(-T,T)}+\|\widetilde{u_{2,z}}-\widetilde{u_{2,z_0}}\|_{\widetilde{H}^1(-T,T)}:\tilde{g}(z)-\tilde{g}(z_0)\\
&=(\widetilde{u_{1,z}}+\widetilde{u_{2,z}})-(\widetilde{u_{1,z_0}}+\widetilde{u_{2,z_0}})\}\\
&=\inf\{\|u_{1,z}-u_{1,z_0}\|_{L^2(0,T)}+\|u_{2,z}-u_{2,z_0}\|_{{}_0H^1(0,T)}:g(z)-g(z_0)\\
&=(u_{1,z}+u_{2,z})-(u_{1,z_0}+u_{2,z_0})\}\\
&=\|g(z)-g(z_0)\|_{L^2(0,T)+\,{}_0H^1(0,T)}.
\end{aligned}$$

Since $g$ is continuous in $S$, we have the following implication:

$$z\to z_0 \Rightarrow \|g(z)-g(z_0)\|_{L^2(0,T)+\,{}_0H^1(0,T)}\to 0.$$

From $\|\tilde{g}(z)-\tilde{g}(z_0)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}=\|g(z)-g(z_0)\|_{L^2(0,T)+\,{}_0H^1(0,T)}$, we deduce that

$$z\to z_0 \Rightarrow \|\tilde{g}(z)-\tilde{g}(z_0)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}\to 0.$$

Therefore, $\tilde{g}$ is continuous in $S$.

Furthermore, we want to prove that $\tilde{g}$ is bounded in $S$. Indeed, we have

$$\begin{aligned}
&\|\tilde{g}(z)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}\\
&=\inf\{\|\widetilde{u_{1,z}}\|_{\widetilde{L}^2(-T,T)}+\|\widetilde{u_{2,z}}\|_{\widetilde{H}^1(-T,T)}:\tilde{g}(z)=\widetilde{u_{1,z}}+\widetilde{u_{2,z}}\}\\
&=\inf\{\|u_{1,z}\|_{L^2(0,T)}+\|u_{2,z}\|_{{}_0H^1(0,T)}:g(z)=u_{1,z}+u_{2,z}\}\\
&=\|g(z)\|_{L^2(0,T)+\,{}_0H^1(0,T)}.
\end{aligned}$$

We know that $g$ is bounded in $S$, so we know that there exists $M>0$ such that

$$\|g(z)\|_{L^2(0,T)+\,{}_0H^1(0,T)}\le M \quad \text{for all } z\in S.$$

From

$$\|\tilde{g}(z)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}=\|g(z)\|_{L^2(0,T)+\,{}_0H^1(0,T)}$$

we deduce that $\|\tilde{g}(z)\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}\le M$ for all $z\in S$. Hence, $\tilde{g}$ is bounded in $S$.

Now, we check if $\tilde{g}$ is analytic in $S^0$. We take $z_0\in S^0$. We know that $g$ is analytic in $S^0$, so there exists $g'(z_0)$, and we have

$$g'(z_0)=\lim_{h\to 0}\frac{g(z_0+h)-g(z_0)}{h} \tag{44}$$

in the norm topology on $L^2(0,T)+\,{}_0H^1(0,T)$. A candidate for $\tilde{g}'(z_0)$ is

$$\widetilde{g'(z_0)}(t):=\begin{cases} g'(z_0)(t) & \text{for } t\in(0,T),\\ 0 & \text{for } t\in(-T,0].\end{cases}$$

We will show that

$$\lim_{h\to 0}\left\|\frac{\tilde{g}(z_0+h)-\tilde{g}(z_0)}{h}-\widetilde{g'(z_0)}\right\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}=0. \tag{45}$$

We can see that

$$\frac{\tilde{g}(z_0+h)-\tilde{g}(z_0)}{h}-\widetilde{g'(z_0)}=\frac{\widetilde{u_{1,z_0+h}}+\widetilde{u_{2,z_0+h}}-(\widetilde{u_{1,z_0}}+\widetilde{u_{2,z_0}})}{h}-(\widetilde{u'_{1,z_0}}+\widetilde{u'_{2,z_0}}).$$

Hence, we obtain

$$\begin{aligned}
&\left\|\frac{\tilde{g}(z_0+h)-\tilde{g}(z_0)}{h}-\widetilde{g'(z_0)}\right\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}\\
&=\inf\left\{\left\|\frac{\widetilde{u_{1,z_0+h}}-\widetilde{u_{1,z_0}}}{h}-\widetilde{u'_{1,z_0}}\right\|_{\widetilde{L}^2(-T,T)}+\left\|\frac{\widetilde{u_{2,z_0+h}}-\widetilde{u_{2,z_0}}}{h}-\widetilde{u'_{2,z_0}}\right\|_{\widetilde{H}^1(-T,T)}:\right.\\
&\left.\frac{\tilde{g}(z_0+h)-\tilde{g}(z_0)}{h}-\widetilde{g'(z_0)}=\frac{\widetilde{u_{1,z_0+h}}+\widetilde{u_{2,z_0+h}}-(\widetilde{u_{1,z_0}}+\widetilde{u_{2,z_0}})}{h}-(\widetilde{u'_{1,z_0}}+\widetilde{u'_{2,z_0}})\right\}\\
&=\inf\left\{\left\|\frac{u_{1,z_0+h}-u_{1,z_0}}{h}-u'_{1,z_0}\right\|_{L^2(0,T)}+\left\|\frac{u_{2,z_0+h}-u_{2,z_0}}{h}-u'_{2,z_0}\right\|_{{}_0H^1(0,T)}:\right.\\
&\left.\frac{g(z_0+h)-g(z_0)}{h}-g'(z_0)=\frac{u_{1,z_0+h}+u_{2,z_0+h}-(u_{1,z_0}+u_{2,z_0})}{h}-(u'_{1,z_0}+u'_{2,z_0})\right\}\\
&=\left\|\frac{g(z_0+h)-g(z_0)}{h}-g'(z_0)\right\|_{L^2(0,T)+{}_0H^1(0,T)}.
\end{aligned}$$

From (44), we have

$$\lim_{h\to 0}\left\|\frac{g(z_0+h)-g(z_0)}{h}-g'(z_0)\right\|_{L^2(0,T)+{}_0H^1(0,T)}=0,$$

so from the above calculations we deduce that

$$\lim_{h\to 0}\left\|\frac{\tilde{g}(z_0+h)-\tilde{g}(z_0)}{h}-\widetilde{g'(z_0)}\right\|_{\widetilde{L}^2(-T,T)+\widetilde{H}^1(-T,T)}=0.$$

Hence, we have (45), and thus we obtain that $\tilde{g}'(z_0)=\widetilde{g'(z_0)}$ for all $z_0\in S^0$, so $\tilde{g}$ is analytic in $S^0$.

Furthermore, we know that $\tilde{g}(it)\in\widetilde{L}^2(-T,T)$ because

$$\|\tilde{g}(it)\|_{\widetilde{L}^2(-T,T)}=\|\widetilde{g(it)}\|_{\widetilde{L}^2(-T,T)}=\|g(it)\|_{L^2(0,T)}.$$

Moreover, we know that the map $t\mapsto g(it)$ is bounded with respect to the space $L^2(0,T)$, so from the equality $\|\tilde{g}(it)\|_{\widetilde{L}^2(-T,T)}=\|g(it)\|_{L^2(0,T)}$ we deduce that $t\mapsto\tilde{g}(it)$ is bounded with respect to the space $\widetilde{L}^2(-T,T)$. Furthermore, we know that the map $t\mapsto g(it)$ is continuous with respect to the space $L^2(0,T)$. It means that if $t \to t_0$, then $\|g(it) - g(it_0)\|_{L^2(0,T)} \to 0$. We have

$$\|\tilde{g}(it) - \tilde{g}(it_0)\|_{\widetilde{L}^2(-T,T)} = \|\widetilde{g(it)} - \widetilde{g(it_0)}\|_{\widetilde{L}^2(-T,T)} = \|g(it) - g(it_0)\|_{L^2(0,T)}.$$

From the above calculations, we obtain the following implication:

$$t \to t_0 \Rightarrow \|\tilde{g}(it) - \tilde{g}(it_0)\|_{\widetilde{L}^2(-T,T)} \to 0,$$

so $t \mapsto \tilde{g}(it)$ is continuous with respect to the space $\widetilde{L}^2(-T,T)$. In the same way, we can show that $\tilde{g}(it+1) \in \widetilde{H}^1(-T,T)$, because

$$\|\tilde{g}(it+1)\|_{\widetilde{H}^1(-T,T)} = \|\widetilde{g(it+1)}\|_{\widetilde{H}^1(-T,T)} = \|g(it+1)\|_{{}_0H^1(0,T)}.$$

Thus, $t \mapsto \tilde{g}(it+1)$ is bounded with respect to the space $\widetilde{H}^1(-T,T)$, because $t \mapsto g(it+1)$ is bounded with respect to the space ${}_0H^1(0,T)$. Moreover, we know that the map $t \mapsto g(it+1)$ is continuous with respect to the space ${}_0H^1(0,T)$. It means that if $t \to t_0$, then $\|g(it+1) - g(it_0+1)\|_{{}_0H^1(0,T)} \to 0$. We have

$$\begin{aligned}\|\tilde{g}(it+1) - \tilde{g}(it_0+1)\|_{\widetilde{H}^1(-T,T)} &= \|\widetilde{g(it+1)} - \widetilde{g(it_0+1)}\|_{\widetilde{H}^1(-T,T)} \\ &= \|g(it+1) - g(it_0+1)\|_{{}_0H^1(0,T)}.\end{aligned}$$

From the above calculations, we obtain the following implication:

$$t \to t_0 \Rightarrow \|\tilde{g}(it+1) - \tilde{g}(it_0+1)\|_{\widetilde{H}^1(-T,T)} \to 0,$$

so $t \mapsto \tilde{g}(it+1)$ is continuous with respect to the space $\widetilde{H}^1(-T,T)$.

As the result, we get $\tilde{g} \in \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T))$. Furthermore,

$$\begin{aligned}\|\tilde{g}\|_{\mathcal{F}(\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T))} &= \max\{\sup_{t\in\mathbb{R}} \|\tilde{g}(it)\|_{\widetilde{L}^2(-T,T)}, \sup_{t\in\mathbb{R}} \|\tilde{g}(1+it)\|_{\widetilde{H}^1(-T,T)}\} \\ &= \max\{\sup_{t\in\mathbb{R}} \|\widetilde{g(it)}\|_{\widetilde{L}^2(-T,T)}, \sup_{t\in\mathbb{R}} \|\widetilde{g(1+it)}\|_{\widetilde{H}^1(-T,T)}\} \\ &= \max\{\sup_{t\in\mathbb{R}} \|g(it)\|_{L^2(0,T)}, \sup_{t\in\mathbb{R}} \|g(1+it)\|_{{}_0H^1(0,T)}\} \\ &= \|g\|_{\mathcal{F}(L^2(0,T),\ {}_0H^1(0,T))}.\end{aligned}$$

Hence, we have (43).

4. Now we are ready to show (42). Take $f \in [L^2(0,T),\ {}_0H^1(0,T)]_\alpha$. We know that $f = g(\alpha)$ for some $g \in \mathcal{F}(L^2(0,T),\ {}_0H^1(0,T))$. Let $\tilde{f} = \tilde{g}(\alpha)$. We will show that

$$f \mapsto \tilde{f}$$

is an isometric isomorphism between $[L^2(0,T),\ {}_0H^1(0,T)]_\alpha$ and $[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$. From 3., we know that $\tilde{g} \in \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T))$, so $\tilde{f} \in [\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$. Furthermore,

$$\|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_\alpha} = \inf\{\|\tilde{g}\|_{\mathcal{F}(\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T))} : \tilde{g}(\alpha) = \tilde{f}\}$$

$$= \inf\{\|g\|_{\mathcal{F}(L^2(0,T),\ {}_0H^1(0,T))} : g(\alpha) = f\} = \|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha}.$$

As the result, we get (42).

$\square$

**Remark 6.** *Let $T > 0$. Then $\widetilde{L}^2(-T,T)$ is a closed subspace of $L^2(-T,T)$.*

*Proof.* Let $(\tilde{f}_n)_{n\in\mathbb{N}} \subset \widetilde{L}^2(-T,T)$ and $\|\cdot\|_{L^2(-T,T)} - \lim_{n\to\infty} \tilde{f}_n = f$. Due to the fact that $L^2(-T,T)$ is complete, we get $f \in L^2(-T,T)$. Take $\varepsilon > 0$. There exists $N_\varepsilon > 0$ such that for all $n \geq N_\varepsilon$ we get

$$\varepsilon > \|\tilde{f}_n - f\|^2_{L^2(-T,T)} = \int_{-T}^{0} |f(t)|^2 dt + \int_0^T |\tilde{f}_n(t) - f(t)|^2 dt \geq \int_{-T}^{0} |f(t)|^2 dt.$$

Hence,

$$0 \leq \int_{-T}^{0} |f(t)|^2 dt < \varepsilon$$

for all $\varepsilon > 0$. Going to the limit as $\varepsilon \to 0$, we obtain

$$\int_{-T}^{0} |f(t)|^2 dt = 0.$$

From the above equality, we deduce that $f = 0$ almost everywhere in $(-T,0)$ and $f \in \widetilde{L}^2(-T,T)$. Otherwise, there exists a set $A \subset (-T,0)$ of Lebesgue measure zero such that $f(t) \neq 0$ for $t \in A$. Hence, $|f(t)| > 0$ for $t \in A$ and there exists $\delta > 0$ such that $\int_A |f(t)|^2 dt > \delta$. Then

$$0 = \int_{-T}^{0} |f(t)|^2 dt = \int_A |f(t)|^2 dt + \int_{(-T,0)\setminus A} |f(t)|^2 dt > \delta,$$

which is a contradiction. $\square$

*Proof of Theorem 6.* We divide our proof into two parts.
1. We show that

$$[L^2(0,T),\ {}_0H^1(0,T)]_\alpha \hookrightarrow {}_0H^\alpha(0,T). \tag{46}$$

Take $f \in [L^2(0,T),\ {}_0H^1(0,T)]_\alpha$. From (42), we can deduce that $\tilde{f} \in [\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$. Now, we would like to show that

$$[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha \hookrightarrow [L^2(-T,T), H^1(-T,T)]_\alpha. \tag{47}$$

From Definition 2, we have

$$\tilde{f} = g(\alpha),$$

where $g \in \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T))$ i.e.

$$g : S = \{z \in \mathbb{C} : 0 \leq Rez \leq 1\} \mapsto \widetilde{L}^2(-T,T) + \widetilde{H}^1(-T,T)$$

and

1. g is continuous and bounded in $S$,
2. g is analytic in $S^0 = \{z \in \mathbb{C} : 0 < Rez < 1\}$,
3. $g(it) \in \widetilde{L}^2(-T,T)$ and $g(it+1) \in \widetilde{H}^1(-T,T)$ for all $t \in \mathbb{R}$,
4. functions $t \mapsto g(it)$ and $t \mapsto g(it+1)$ are bounded and continuous with respect to the spaces $\widetilde{L}^2(-T,T)$ and $\widetilde{H}^1(-T,T)$, respectively.

Due to the fact that $\widetilde{L}^2(-T,T) \hookrightarrow L^2(-T,T)$ and $\widetilde{H}^1(-T,T) \hookrightarrow H^1(-T,T)$, we can write

$$g : S = \{z \in \mathbb{C} : 0 \leq Rez \leq 1\} \mapsto L^2(-T,T) + H^1(-T,T)$$

and

1. g is continuous and bounded in $S$,
2. g is analytic in $S^0 = \{z \in \mathbb{C} : 0 < Rez < 1\}$,
3. $g(it) \in L^2(-T,T)$ and $g(it+1) \in H^1(-T,T)$ for all $t \in \mathbb{R}$,
4. functions $t \mapsto g(it)$ and $t \mapsto g(it+1)$ are bounded and continuous with respect to the spaces $L^2(-T,T)$ and $H^1(-T,T)$, respectively.

Therefore, we get $g \in \mathcal{F}(L^2(-T,T), H^1(-T,T))$ and $\tilde{f} = g(\alpha)$. So, from Definition 2, we can deduce that $\tilde{f} \in [L^2(-T,T), H^1(-T,T)]_\alpha$. In this way, we get an algebraic inclusion

$$[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha \subseteq [L^2(-T,T), H^1(-T,T)]_\alpha.$$

However, we still must show that $[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$ is continuously embedded in $[L^2(-T,T), H^1(-T,T)]_\alpha$. Indeed,

$$\begin{aligned}
\|g\|_{\mathcal{F}(L^2(-T,T),H^1(-T,T))} &= \max\{\sup_{t\in\mathbb{R}} \|g(it)\|_{L^2(-T,T)}, \sup_{t\in\mathbb{R}} \|g(1+it)\|_{H^1(-T,T)}\} \\
&\leq \max\{\sup_{t\in\mathbb{R}} \|g(it)\|_{\widetilde{L}^2(-T,T)}, \sup_{t\in\mathbb{R}} \|g(1+it)\|_{\widetilde{H}^1(-T,T)}\} \\
&= \|g\|_{\mathcal{F}(\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T))}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\|\tilde{f}\|_{[L^2(-T,T),H^1(-T,T)]_\alpha} \\
&= \inf\{\|g\|_{\mathcal{F}(L^2(-T,T),H^1(-T,T))} : g \in \mathcal{F}(L^2(-T,T), H^1(-T,T)),\ g(\alpha) = \tilde{f}\} \\
&\leq \inf\{\|g\|_{\mathcal{F}(L^2(-T,T),H^1(-T,T))} : g \in \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)),\ g(\alpha) = f\} \\
&\leq \inf\{\|g\|_{\mathcal{F}(\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T))} : g \in \mathcal{F}(\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)),\ g(\alpha) = \tilde{f}\} \\
&= \|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_\alpha}.
\end{aligned}$$

Then we get (47). Furthermore, from Theorem 30 we know that

$$[L^2(-T,T),H^1(-T,T)]_\alpha = H^\alpha(-T,T) \tag{48}$$

with equivalent norms. Thus,

$$\tilde{f} \in H^\alpha(-T,T) \tag{49}$$

and

$$\begin{aligned} \|\tilde{f}\|_{H^\alpha(-T,T)} &\le C\|\tilde{f}\|_{[L^2(-T,T),H^1(-T,T)]_\alpha} \le C\|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_\alpha} \\ &= C\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha}. \end{aligned} \tag{50}$$

Due to the fact that $\tilde{f} \in H^\alpha(-T,T)$, using Definition 3, we have $\tilde{f} \in L^2(-T,T)$ and

$$\int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2}dxdy < +\infty.$$

Then we get $f \in L^2(0,T)$ and

$$\begin{aligned} \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy &= \int_{-T}^{0}\int_{-T}^{T} \frac{|\tilde{f}(x)|^2}{|x-y|^{1+2\alpha}}dxdy + \int_{0}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\ &= \int_{-T}^{0}\int_{0}^{T} \frac{|f(x)|^2}{|x-y|^{1+2\alpha}}dxdy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\ &\quad + \int_{0}^{T}\int_{-T}^{0} \frac{|f(y)|^2}{|x-y|^{1+2\alpha}}dxdy = 2\int_{0}^{T}\int_{-T}^{0} \frac{|f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\ &\quad + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy. \end{aligned}$$

We can notice that

$$\int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \le \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy < +\infty.$$

Thus, $f \in H^\alpha(0,T)$ and

$$\begin{aligned} \|f\|^2_{H^\alpha(0,T)} &= \|f\|^2_{L^2(0,T)} + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\ &\le \|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy = \|\tilde{f}\|^2_{H^\alpha(-T,T)}. \end{aligned}$$

Using (50), we obtain

$$\|f\|_{H^\alpha(0,T)} \le C\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha}.$$

Therefore,

$$[L^2(0,T),\ {}_0H^1(0,T)]_\alpha \hookrightarrow H^\alpha(0,T). \tag{51}$$

We notice that for $\alpha \neq \frac{1}{2}$, we have

$$\|f\|_{{}_0H^\alpha(0,T)} = \|f\|_{H^\alpha(0,T)} \leq C\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha}.$$

Hence, for $\alpha \in (0,\frac{1}{2})$ we have (46). Now, we do some calculations in the case when $\alpha = \frac{1}{2}$.

$$\begin{aligned}
&\int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy \\
&= \int_{-T}^{0}\int_{-T}^{T} \frac{|\tilde{f}(x)|^2}{|x-y|^2} dxdy + \int_{0}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy \\
&= \int_{-T}^{0}\int_{0}^{T} \frac{|f(x)|^2}{|x-y|^2} dxdy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy + \int_{0}^{T}\int_{-T}^{0} \frac{|f(y)|^2}{|x-y|^2} dxdy \\
&= 2\int_{0}^{T}\int_{-T}^{0} \frac{|f(y)|^2}{|x-y|^2} dxdy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy \\
&= 2\int_{0}^{T} |f(y)|^2 \int_{-T}^{0} |x-y|^{-2} dxdy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy \\
&= 2\int_{0}^{T} |f(y)|^2 \int_{-T}^{0} (y-x)^{-2} dxdy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy \\
&= 2\int_{0}^{T} \frac{|f(y)|^2}{y} dy - 2\int_{0}^{T} \frac{|f(y)|^2}{y+T} dy + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy.
\end{aligned}$$

Therefore,

$$\int_{0}^{T} \frac{|f(y)|^2}{y} dy \leq \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy + \frac{2}{T}\|f\|^2_{L^2(0,T)} < +\infty$$

because $\tilde{f} \in H^{\frac{1}{2}}(-T,T)$ and $f \in L^2(0,T)$.

As a result, for $\alpha = \frac{1}{2}$ we get

$$f \in H^{\frac{1}{2}}(0,T) \quad \text{and} \quad \int_{0}^{T} \frac{|f(y)|^2}{y} dy < +\infty. \tag{52}$$

Thus, for $\alpha = \frac{1}{2}$ we have $f \in {}_0H^{\frac{1}{2}}(0,T)$. Moreover,

$$\begin{aligned}
\|f\|^2_{{}_0H^{\frac{1}{2}}(0,T)} &= \|f\|^2_{H^{\frac{1}{2}}(0,T)} + \int_{0}^{T} \frac{|f(y)|^2}{y} dy \\
&= \|f\|^2_{L^2(0,T)} + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy + \int_{0}^{T} \frac{|f(y)|^2}{y} dy \\
&= \|\tilde{f}\|^2_{L^2(-T,T)} + \int_{0}^{T}\int_{0}^{T} \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy + \int_{0}^{T} \frac{|f(y)|^2}{y} dy \\
&\leq \|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy + \frac{2}{T}\|f\|^2_{L^2(0,T)}
\end{aligned}$$

$$
\begin{aligned}
&= (1+\frac{2}{T})\|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T}\frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2}dxdy \\
&\le (1+\frac{2}{T})\left(\|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T}\frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2}dxdy\right) \\
&= (1+\frac{2}{T})\|\tilde{f}\|^2_{H^{\frac{1}{2}}(-T,T)}.
\end{aligned}
$$

Thus, using (50) again, we have

$$
\|f\|_{{}_0H^{\frac{1}{2}}(0,T)} \le C\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_{\frac{1}{2}}}.
$$

Hence, we obtain (46) for $\alpha = \frac{1}{2}$.
It remains to consider the case $\alpha \in (\frac{1}{2},1)$. We have to show that if $f \in [L^2(0,T),\ {}_0H^1(0,T)]_\alpha$ for $\alpha \in (\frac{1}{2},1)$, then $f(0)=0$. Indeed, from Proposition 25 we know that if $f \in [L^2(0,T),\ {}_0H^1(0,T)]_\alpha$, then $\tilde{f} \in [\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha$. Moreover, from (49) and (50) we have

$$
[\widetilde{L}^2(-T,T), \widetilde{H}^1(-T,T)]_\alpha \hookrightarrow H^\alpha(-T,T),
$$

and hence $\tilde{f} \in H^\alpha(-T,T)$. From Chapter 1.4.4 in [3], we know that

$$
H^\alpha(-T,T) \hookrightarrow C^{0,\alpha-\frac{1}{2}}([-T,T]) \quad \text{for } \alpha \in \left(\frac{1}{2},1\right).
$$

Hence, if $\tilde{f} \in H^\alpha(-T,T)$, then there exists a function $\tilde{f}^* \in C^{0,\alpha-\frac{1}{2}}([-T,T])$ such that $\tilde{f} = \tilde{f}^*$ almost everywhere. Hence, function $\tilde{f}$ has a continuous representative $\tilde{f}^*$, which is defined on $[-T,T]$. Furthermore, $\tilde{f}^*(t) = 0$ for $t \in (-T,0)$. Since $\tilde{f}^*$ is continuous on $[-T,T]$, we get $\tilde{f}^*(0) = 0$. Hence, $\tilde{f}(0) = 0$ in a trace sense and $f = \tilde{f}_{|(0,T)}$, so we deduce that $f(0) = 0$. Therefore, (51) implies that

$$
[L^2(0,T),\ {}_0H^1(0,T)]_\alpha \hookrightarrow\ {}_0H^\alpha(0,T)
$$

for $\alpha \in (\frac{1}{2},1)$. Thus, we get (51) for $\alpha \in (\frac{1}{2},1)$. In our previous considerations, we proved that (51) holds for $\alpha \in (0,\frac{1}{2}]$. Hence, we get (51) for all $\alpha \in (0,1)$, and it finishes this part of the proof.
2. We will show that

$$
{}_0H^\alpha(0,T) \hookrightarrow [L^2(0,T),\ {}_0H^1(0,T)]_\alpha. \tag{53}
$$

Take $f \in\ {}_0H^\alpha(0,T)$, and define

$$
\tilde{f}(t) = \begin{cases} f(t) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0). \end{cases}
$$

- First we show that $\tilde{f} \in H^\alpha(-T,T)$. Notice that ${}_0H^\alpha(0,T)$ is a subspace of $H^\alpha(0,T)$. Thus, $f \in H^\alpha(0,T)$. From Definition 3, we deduce that

$$
f \in L^2(0,T) \quad \text{and} \quad \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy < +\infty. \tag{54}
$$

Therefore, $\tilde{f} \in L^2(-T,T)$, because $\|\tilde{f}\|_{L^2(-T,T)} = \|f\|_{L^2(0,T)}$. In order to show that $\tilde{f} \in H^\alpha(-T,T)$, it remains to verify if

$$\int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}} dxdy < +\infty. \tag{55}$$

First, we show (55) for $\alpha = \frac{1}{2}$. Using the calculations from the first part of the proof, we obtain

$$\begin{aligned}
&\int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy \\
&= 2\int_0^T \frac{|f(y)|^2}{y} dy - 2\int_0^T \frac{|f(y)|^2}{y+T} dy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy \\
&\le 2\int_0^T \frac{|f(y)|^2}{y} dy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^2} dxdy \le 2\|f\|^2_{{}_0H^{\frac{1}{2}}(0,T)} < +\infty
\end{aligned}$$

because $f \in {}_0H^{\frac{1}{2}}(0,T)$. Thus, we get (55) for $\alpha = \frac{1}{2}$, and hence $\tilde{f} \in H^{\frac{1}{2}}(-T,T)$. Moreover, we get the following estimate:

$$\begin{aligned}
\|\tilde{f}\|^2_{H^{\frac{1}{2}}(-T,T)} &= \|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy \\
&= \|f\|^2_{L^2(0,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^2} dxdy \\
&\le \|f\|^2_{L^2(0,T)} + 2\|f\|^2_{{}_0H^{\frac{1}{2}}(0,T)} \le 3\|f\|^2_{{}_0H^{\frac{1}{2}}(0,T)}.
\end{aligned}$$

Thus, we can write

$$\|\tilde{f}\|_{H^{\frac{1}{2}}(-T,T)} \le \sqrt{3}\|f\|_{{}_0H^{\frac{1}{2}}(0,T)}. \tag{56}$$

Now, we show (55) for $\alpha \in (0,1)$, but $\alpha \neq \frac{1}{2}$. Using the similar calculations as in the case when $\alpha = \frac{1}{2}$, we get

$$\int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}} dxdy \le 2\int_0^T \frac{|f(y)|^2}{y^{2\alpha}} dy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}} dxdy.$$

Again, the second integral is finite due to the (54). We use Lemma 24, in order to show that $\int_0^T \frac{|f(y)|^2}{y^{2\alpha}} dy < +\infty$. We know that $f \in {}_0H^\alpha(0,T)$, so $f \in H^\alpha(0,T)$ for $\alpha \in (0,\frac{1}{2})$ and $f \in \{u \in H^\alpha(0,T) : u(0) = 0\}$ for $\alpha \in (\frac{1}{2},1)$. From Theorem 20, we know that $H_0^\alpha(0,T) = H^\alpha(0,T)$ for $\alpha \in (0,\frac{1}{2})$. Thus, we can use Lemma 24, and we get

$$\int_0^T \frac{|f(y)|^2}{y^{2\alpha}} dy \le \int_0^T \frac{|f(y)|^2}{[\mathrm{dist}(y,\{0,T\})]^{2\alpha}} dy \le C(s,p,\sigma)\|f\|^2_{H^\alpha(0,T)} \tag{57}$$

for $\alpha \in (0,\frac{1}{2})$. Moreover, for $\alpha \in (\frac{1}{2},1)$ we define

$$F(t) = \begin{cases} f(t) & \text{if } t \in (0,T), \\ f(2T-t) & \text{if } t \in (T,2T). \end{cases}$$

We have $F \in H^{\alpha}(0,2T)$, because

$$
\begin{aligned}
\|F\|^2_{H^\alpha(0,2T)} &= \|F\|^2_{L^2(0,2T)} + \int_0^{2T}\int_0^{2T} \frac{|F(x)-F(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&= \int_0^T |f(x)|^2dx + \int_T^{2T} |f(2T-x)|^2dx + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^T\int_T^{2T} \frac{|f(2T-x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_T^{2T}\int_0^{T} \frac{|f(x)-f(2T-y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_T^{2T}\int_T^{2T} \frac{|f(2T-x)-f(2T-y)|^2}{|x-y|^{1+2\alpha}}dxdy = 2\int_0^T |f(x)|^2dx \\
&\quad + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|2T-x-y|^{1+2\alpha}}dxdy \\
&\quad + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-2T+y|^{1+2\alpha}}dxdy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\le 2\|f\|^2_{L^2(0,T)} + 4\int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy \le 4\|f\|^2_{H^\alpha(0,T)},
\end{aligned}
$$

where we used the fact that $|2T-x-y| \ge |x-y|$ and $|x-2T+y| \ge |x-y|$ for all $x,y \in (0,T)$. Furthermore, $F(0) = F(2T) = f(0) = 0$. Hence, $F \in H^{\alpha}_0(0,2T)$ for $\alpha \in (\frac{1}{2},1)$. Therefore, we can use Lemma 24 for function $F$, and we get

$$
\begin{aligned}
\int_0^T \frac{|f(y)|^2}{y^{2\alpha}}dy &\le \int_0^{2T} \frac{|F(y)|^2}{y^{2\alpha}}dy \le \int_0^{2T} \frac{|F(y)|^2}{[\mathrm{dist}(y,\{0,2T\})]^{2\alpha}}dy \\
&\le C(s,p,\sigma)\|F\|^2_{H^\alpha(0,2T)} \le C(s,p,\sigma)\|f\|^2_{H^\alpha(0,T)}
\end{aligned} \tag{58}
$$

for $\alpha \in (\frac{1}{2},1)$. Thus, from (57) and (58) we obtain

$$
\int_0^T \frac{|f(y)|^2}{y^{2\alpha}}dy \le C(s,p,\sigma)\|f\|^2_{H^\alpha(0,T)}
$$

for $\alpha \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Hence, we get (55) for $\alpha \in (0,1)$, but $\alpha \ne \frac{1}{2}$, and then $\tilde{f} \in H^{\alpha}(-T,T)$ for $\alpha \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Moreover, from Lemma 24 we obtain the following estimate:

$$
\begin{aligned}
\|\tilde{f}\|^2_{H^\alpha(-T,T)} &= \|\tilde{f}\|^2_{L^2(-T,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&= \|f\|^2_{L^2(0,T)} + \int_{-T}^{T}\int_{-T}^{T} \frac{|\tilde{f}(x)-\tilde{f}(y)|^2}{|x-y|^{1+2\alpha}}dxdy \\
&\le \|f\|^2_{L^2(0,T)} + 2\int_0^T \frac{|f(y)|^2}{y^{2\alpha}}dy + \int_0^T\int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}}dxdy
\end{aligned}
$$

$$\leq \|f\|^2_{L^2(0,T)} + 2C(s,p,\sigma)\|f\|^2_{H^\alpha(0,T)} + \int_0^T \int_0^T \frac{|f(x)-f(y)|^2}{|x-y|^{1+2\alpha}} dxdy$$
$$= (1+2C(s,p,\sigma))\|f\|^2_{H^\alpha(0,T)} = (1+2C(s,p,\sigma))\|f\|^2_{{}_0H^\alpha(0,T)}$$

for $\alpha \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Therefore, from the above calculations and from (56) we have

$$\|\tilde{f}\|_{H^\alpha(-T,T)} \leq C\|f\|_{{}_0H^\alpha(0,T)} \tag{59}$$

for $\alpha \in (0,1)$.

- From Theorem 30 we get

$$H^\alpha(-T,T) = [L^2(-T,T), H^1(-T,T)]_\alpha$$

with equivalence of the respective norms. Therefore, we have $\tilde{f} \in [L^2(-T,T), H^1(-T,T)]_\alpha$, and we obtain the following estimate:

$$\|\tilde{f}\|_{[L^2(-T,T),H^1(-T,T)]_\alpha} \leq C\|\tilde{f}\|_{H^\alpha(-T,T)}. \tag{60}$$

We recall that the right translation semigroup $(G_r(t))_{t\geq 0}$ is defined in Definition 13. It follows from Proposition 14, that $A = -\frac{d}{dx}$ is the infinitesimal generator of the right translation semigroup $(G_r(t))_{t\geq 0}$ , which is defined on the space $L^2(-T,T)$. Moreover, from Proposition 14 we know that $D(A) = H^1(-T,T)$. Therefore, from Theorem 15 we get that if

$$\tilde{f} \in [L^2(-T,T), H^1(-T,T)]_\alpha = [H^1(-T,T), L^2(-T,T)]_{1-\alpha},$$

then

$$t^{-\frac{1}{2}-\alpha}(G_r(t)\tilde{f} - \tilde{f}) \in L^2(0,\infty, L^2(-T,T)).$$

Furthermore, from Theorem 15 we obtain the estimate

$$\left(\|\tilde{f}\|^2_{L^2(-T,T)} + \int_0^\infty t^{2(-\frac{1}{2}-\alpha)}\|G_r(t)\tilde{f} - \tilde{f}\|^2_{L^2(-T,T)} dt\right)^{\frac{1}{2}} \leq C\|\tilde{f}\|_{[L^2(-T,T),H^1(-T,T)]_\alpha}. \tag{61}$$

Further, from Definition 13, we obtain

$$t^{-\frac{1}{2}-\alpha}(G_r(t)\tilde{f}(x) - \tilde{f}(x)) = 0$$

for $x \in (-T,0)$ and $t \geq 0$. Hence, we have

$$t^{-\frac{1}{2}-\alpha}(G_r(t)\tilde{f} - \tilde{f}) \in L^2(0,\infty, \widetilde{L}^2(-T,T)). \tag{62}$$

We define a new family of operators $(\tilde{G}_r(t))_{t\geq 0}$ such that for all $t \geq 0$

$$\tilde{G}_r(t) : \widetilde{L}^2(-T,T) \to \widetilde{L}^2(-T,T)$$

and

$$\tilde{G}_r(t)\tilde{h} := G_r(t)\tilde{h},$$

where $\tilde{h} \in \widetilde{L}^2(-T,T)$. We want to use Theorem 15 for $X = \widetilde{H}^1(-T,T), Y = \widetilde{L}^2(-T,T)$ and for $(\tilde{G}_r(t))_{t\geq 0}$. To this purpose, we must verify if $(\tilde{G}_r(t))_{t\geq 0}$ satisfies (10) i.e., if it is a continuous semigroup.

Let $\tilde{h} \in \widetilde{L}^2(-T,T)$. We have
1.

$$(\tilde{G}_r(0)\tilde{h})(x) = \begin{cases} \tilde{h}(x) & \text{if } x > -T, \\ 0 & \text{if } x < -T, \end{cases}$$

but we always take $x \in (-T,T)$, so $(\tilde{G}_r(0)\tilde{h})(x) = \tilde{h}(x)$ for $x \in (-T,T)$. Hence, $\tilde{G}_r(0) = I$.
2. Let $t,s \geq 0$. We get

$$\begin{aligned} (\tilde{G}_r(t)\tilde{G}_r(s)\tilde{h})(x) &= \begin{cases} (\tilde{G}_r(s)\tilde{h})(x-t) & \text{if } x-t > -T, \\ 0 & \text{if } x-t < -T \end{cases} \\ &= \begin{cases} \tilde{h}(x-t-s) & \text{if } x-t-s > -T, \\ 0 & \text{if } x-t-s < -T \end{cases} \\ &= (\tilde{G}_r(t+s)\tilde{h})(x). \end{aligned}$$

3. We know that for all $t \geq 0$ $G_r(t) \in \mathbf{B}(L^2(-T,T), L^2(-T,T))$. We want to show that $\tilde{G}_r(t) \in \mathbf{B}(\widetilde{L}^2(-T,T), \widetilde{L}^2(-T,T))$. Let $t \geq 0$. We can easy see that for $\tilde{h} \in \widetilde{L}^2(-T,T)$ we have

$$(\tilde{G}_r(t)\tilde{h})(x) = \begin{cases} h(x-t) & \text{if } 0 < x-t < T, \\ 0 & \text{if } x-t < 0, \end{cases}$$

so $\tilde{G}_r(t)\tilde{h} \in \widetilde{L}^2(-T,T)$. Further,

$$\begin{aligned} &\|\tilde{G}_r(t)\|_{\mathbf{B}(\widetilde{L}^2(-T,T),\widetilde{L}^2(-T,T))} \\ &= \sup\{\|\tilde{G}_r(t)\tilde{h}\|_{L^2(-T,T)} : \tilde{h} \in \widetilde{L}^2(-T,T),\ \|\tilde{h}\|_{L^2(-T,T)} \leq 1\} \\ &= \sup\{\|G_r(t)\tilde{h}\|_{L^2(-T,T)} : \tilde{h} \in \widetilde{L}^2(-T,T),\ \|\tilde{h}\|_{L^2(-T,T)} \leq 1\} \\ &\leq \sup\{\|G_r(t)h\|_{L^2(-T,T)} : h \in L^2(-T,T),\ \|h\|_{L^2(-T,T)} \leq 1\} \\ &= \|G_r(t)\|_{\mathbf{B}(L^2(-T,T),L^2(-T,T))}, \end{aligned}$$

so we have $\tilde{G}_r(t) \in \mathbf{B}(\widetilde{L}^2(-T,T), \widetilde{L}^2(-T,T))$.
4. We know that

$$\forall h \in L^2(-T,T), \quad \|G_r(t)h - h\|_{L^2(-T,T)} \to 0 \quad \text{as } t \downarrow 0.$$

We would like to show that

$$\forall \tilde{h} \in \widetilde{L}^2(-T,T), \quad \|\tilde{G}_r(t)\tilde{h} - \tilde{h}\|_{L^2(-T,T)} \to 0 \quad \text{as } t \downarrow 0. \tag{63}$$

We take $\tilde{h} \in \widetilde{L}^2(-T,T)$ and let $t_n \to 0$ as $n \to +\infty$. Then we have

$$\|\tilde{G}_r(t_n)\tilde{h} - \tilde{h}\|_{L^2(-T,T)} = \|G_r(t_n)\tilde{h} - \tilde{h}\|_{L^2(-T,T)} \to 0 \text{ as } n \to +\infty.$$

Hence, we get (63). Moreover, from $1.-4.$ we deduce that $(\tilde{G}_r(t))_{t\geq 0}$ satisfies (10).

- Now, we would like to verify if $(\tilde{G}_r(t))_{t\geq 0}$ satisfies (11). By $\tilde{A}$ we denote the infinitesimal generator of the semigroup $(\tilde{G}_r(t))_{t\geq 0}$. From Definition 9 we have

$$\tilde{A}\tilde{h} := \lim_{t\downarrow 0} \frac{\tilde{G}_r(t)\tilde{h}-\tilde{h}}{t} \in \widetilde{L}^2(-T,T) \quad \text{for} \quad \tilde{h}\in D(\tilde{A}),$$

where

$$D(\tilde{A}) = \left\{ \tilde{h}\in \widetilde{L}^2(-T,T) : \lim_{t\downarrow 0} \frac{\tilde{G}_r(t)\tilde{h}-\tilde{h}}{t} \text{ exists in } \widetilde{L}^2(-T,T) \right\}.$$

So $\tilde{A} : D(\tilde{A}) \to \widetilde{L}^2(-T,T)$. Further, reasoning in exactly the same way as in the proof of Proposition 14 (see Proposition 33 in Appendix), we get

$$\tilde{A}\tilde{h} := -\tilde{h}'$$

with domain:

$$D(\tilde{A}) = \{\tilde{h}\in \widetilde{L}^2(-T,T) : \tilde{h} \text{ absolutely continuous, and } \tilde{h}'\in \widetilde{L}^2(-T,T)\} = \widetilde{H}^1(-T,T).$$

Hence, $(\tilde{G}_r(t))_{t\geq 0}$ satisfies (11).

Due to the observation that $(\tilde{G}_r(t))_{t\geq 0}$ satisfies (10) and (11), we can use Theorem 15 with the semigroup $(\tilde{G}_r(t))_{t\geq 0}$. Moreover, from (62) we get

$$t^{-\frac{1}{2}-\alpha}(\tilde{G}_r(t)\tilde{f}-\tilde{f}) \in L^2(0,\infty;\widetilde{L}^2(-T,T)).$$

Thus, from Theorem 15 we obtain that for $\alpha\in(0,1)$

$$\tilde{f}\in [\widetilde{H}^1(-T,T),\widetilde{L}^2(-T,T)]_{1-\alpha} = [\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_{\alpha}$$

and

$$\|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_{\alpha}} \leq C\left( \|\tilde{f}\|^2_{\widetilde{L}^2(-T,T)} + \int_0^\infty t^{2(-\frac{1}{2}-\alpha)} \|\tilde{G}_r(t)\tilde{f}-\tilde{f}\|^2_{\widetilde{L}^2(-T,T)} dt \right)^{\frac{1}{2}}. \tag{64}$$

However, we observe that

$$\left( \|\tilde{f}\|^2_{\widetilde{L}^2(-T,T)} + \int_0^\infty t^{2(-\frac{1}{2}-\alpha)} \|\tilde{G}_r(t)\tilde{f}-\tilde{f}\|^2_{\widetilde{L}^2(-T,T)} dt \right)^{\frac{1}{2}}$$

$$= \left( \|\tilde{f}\|^2_{L^2(-T,T)} + \int_0^\infty t^{2(-\frac{1}{2}-\alpha)} \|G_r(t)\tilde{f}-\tilde{f}\|^2_{L^2(-T,T)} dt \right)^{\frac{1}{2}}.$$

Hence, from above equality and from (61) and (64), we get

$$\|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_{\alpha}} \leq C\|\tilde{f}\|_{[L^2(-T,T),H^1(-T,T)]_{\alpha}}. \tag{65}$$

Using Theorem 25, we get

$$f \in [L^2(0,T),\ {}_0H^1(0,T)]_\alpha$$

and we have

$$\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha} = \|\tilde{f}\|_{[\widetilde{L}^2(-T,T),\widetilde{H}^1(-T,T)]_\alpha}. \tag{66}$$

Thus, from (59), (60), (65) and (66), we get

$$\|f\|_{[L^2(0,T),\ {}_0H^1(0,T)]_\alpha} \le C\|f\|_{{}_0H^\alpha(0,T)}. \tag{67}$$

Hence, we have (53).

□

# 7. APPENDIX

**Definition 26** (Young's convolution inequality)**.** *Suppose that* $1 \le p \le q \le \infty$ *and* $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$. *Then for any functions* $f \in L^p(\mathbb{R}^n)$ *and* $g \in L^r(\mathbb{R}^n)$, *the function* $f * g$ *is defined almost everywhere (everywhere if* $q = \infty$*), belongs to* $L^q(\mathbb{R}^n)$ *and* $f * g = g * f$ *almost everywhere and*

$$\|f * g\|_{L^q(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^r(\mathbb{R}^n)}.$$

**Definition 27** (Def. 3.1. in [7])**.** *Let* $I \subset \mathbb{R}$ *be an interval. A function* $u : I \to \mathbb{R}$ *is said to be* **absolutely continuous** *on* $I$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sum_{k=1}^{l} |u(b_k) - u(a_k)| \le \varepsilon$$

*for every finite number of nonoverlapping intervals* $(a_k, b_k), k = 1, \ldots, l$, *with* $[a_k, b_k] \subset I$ *and*

$$\sum_{k=1}^{l} |b_k - a_k| \le \delta.$$

*The space of all absolutely continuous functions* $u : I \to \mathbb{R}$ *is denoted by* $AC(I)$.

**Theorem 28** (Thr. 3.30 in [7])**.** *Let* $I \subset \mathbb{R}$ *be an interval. A function* $u : I \to \mathbb{R}$ *belongs to* $AC_{\mathrm{loc}}(I)$ *if and only if*
*(i)* $u$ *is continuous in* $I$,
*(ii)* $u$ *is differentiable* $\mathcal{L}^1-$ *a.e. in* $I$, *and* $u'$ *belongs to* $L^1_{\mathrm{loc}}(I)$,
*(iii) the fundamental theorem of calculus is valid; that is, for all* $x, x_0 \in I$,

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(t)dt.$$

**Example 7.1** (Example 1.8. in [10])**.** *For* $0 < \theta < 1$, $1 \le p < \infty$ *we have*

$$(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{\theta,p} = W^{\theta,p}(\mathbb{R}^n)$$

*with equivalence of the respective norms.*

**Definition 29** (Chapter 1, Section 7.1 in [9])**.** *For* $s \in \mathbb{R}$, *we define*

$$H^s(\mathbb{R}^n) = \{v : v \in \mathcal{S}'(\mathbb{R}^n), (1+|y|^2)^{\frac{s}{2}}\hat{v} \in L^2(\mathbb{R}^n)\},$$

*and provide it with the norm*

$$\|v\|_{H^s(\mathbb{R}^n)} = \|(1+|y|^2)^{\frac{s}{2}}\hat{v}\|_{L^2(\mathbb{R}^n)},$$

*which makes it a Hilbert space.*

**Theorem 30** (Chapter 9, Section 9.1 in [9])**.** *For* $s \in \mathbb{R}$

$$H^s(\Omega) = [L^2(\Omega), H^m(\Omega)]_\theta, \quad \theta m = m - s, \quad m \text{ integer}, \quad 0 < \theta < 1. \tag{68}$$

**Theorem 31** (Thr. 9.1. in [9])**.** *Assume that* $\Omega$ *is an open and bounded set in* $\mathbb{R}^n$ *with smooth boundary. Then* $H^s(\Omega)$ *coincides (algebraically) with the space of restrictions to* $\Omega$ *of the elements of* $H^s(\mathbb{R}^n)$*.*

**Theorem 32** (Thr. 9.2. in [9])**.** *The norm of* $H^s(\Omega)$ *defined by* (68) *is equivalent to the norm*

$$\|u\|_{H^s(\Omega)} = inf\|U\|_{H^s(\mathbb{R}^n)}, \quad U \in H^s(\mathbb{R}^n), \quad U = u \quad a.e. \text{ on } \Omega.$$

**Proposition 33.** *The generator of the semigroup* $(\tilde{G}_r(t))_{t\ge 0}$ *on the space* $X := \widetilde{L}^2(-T,T)$ *is given by*

$$\tilde{A}f := -f'$$

*with the domain*

$$D(\tilde{A}) = \{f \in \widetilde{L}^2(-T,T) : f \text{ absolutely continuous and } f' \in \widetilde{L}^2(-T,T)\}.$$

*Proof.* Let $\tilde{B} : D(\tilde{B}) \to \widetilde{L}^2(-T,T)$ be the infinitesimal generator of the semigroup $(\tilde{G}_r(t))_{t\ge 0}$. We want to show that $\tilde{B} = \tilde{A}$.
1. In the first step, we will show that $\tilde{B} \subset \tilde{A}$. Take $\tilde{f} \in D(\tilde{B})$. From the above observation, we know that

$$\tilde{B}\tilde{f} = \lim_{t\downarrow 0} \frac{\tilde{G}_r(t)\tilde{f} - \tilde{f}}{t} \in \widetilde{L}^2(-T,T). \tag{69}$$

Let $c,d \in (-T,T)$. We have $L^2(-T,T) \hookrightarrow L^2(c,d) \hookrightarrow L^1(c,d)$, and hence

$$\begin{aligned}\left| \int_c^d \frac{\tilde{G}_r(t)\tilde{f}(x) - \tilde{f}(x)}{t} dx - \int_c^d \tilde{B}\tilde{f}(x)dx \right| &\le \int_c^d \left| \frac{\tilde{G}_r(t)\tilde{f}(x) - \tilde{f}(x)}{t} - \tilde{B}\tilde{f}(x) \right| dx \\ &= \left\| \frac{\tilde{G}_r(t)\tilde{f} - \tilde{f}}{t} - \tilde{B}\tilde{f} \right\|_{L^1(c,d)}\end{aligned}$$

$$\leq C \left\| \frac{\tilde{G}_r(t)\tilde{f} - \tilde{f}}{t} - \tilde{B}\tilde{f} \right\|_{L^2(-T,T)}.$$

From (69), we know that the right hand side of the above inequality converges to 0 as $t \downarrow 0$. Thus, we have

$$\int_c^d \frac{\tilde{G}_r(t)\tilde{f}(x) - \tilde{f}(x)}{t} dx \overset{t\to 0^+}{\to} \int_c^d \tilde{B}\tilde{f}(x)dx. \tag{70}$$

We have $t \to 0^+$, so we can assume that $t$ is so small that $c - t > -T$. Due to this observation, we can write

$$\begin{aligned}
\int_c^d \frac{\tilde{G}_r(t)\tilde{f}(x) - \tilde{f}(x)}{t} dx &= \frac{1}{t}\int_c^d \tilde{G}_r(t)\tilde{f}(x)dx - \frac{1}{t}\int_c^d \tilde{f}(x)dx \\
&= \frac{1}{t}\int_{c-t}^{d-t} \tilde{f}(x)dx - \frac{1}{t}\int_c^d \tilde{f}(x)dx \\
&= -\frac{1}{t}\int_{d-t}^{d} \tilde{f}(x)dx + \frac{1}{t}\int_{c-t}^{c} \tilde{f}(x)dx.
\end{aligned}$$

Using the Lebesgue Differentiation Theorem, we get

$$-\frac{1}{t}\int_{d-t}^{d} \tilde{f}(x)dx + \frac{1}{t}\int_{c-t}^{c} \tilde{f}(x)dx \overset{t\to 0^+}{\to} -\tilde{f}(d) + \tilde{f}(c) \quad \text{for a.e. } c,d \in (-T,T). \tag{71}$$

From (70) and (71), we have

$$\tilde{f}(d) = \tilde{f}(c) + \int_c^d (-\tilde{B}\tilde{f})(x)dx \quad \text{for a.e. } c,d \in (-T,T).$$

We set $c_0 \in (a,b)$ such that

$$\tilde{f}(d) = \tilde{f}(c_0) + \int_{c_0}^d (-\tilde{B}\tilde{f})(x)dx \quad \text{for a.e. } d \in (-T,T).$$

Let

$$\tilde{\tilde{f}}(d) := \tilde{f}(c_0) + \int_{c_0}^d (-\tilde{B}\tilde{f})(x)dx \quad \text{for all } d \in (-T,T).$$

Then, we have $\tilde{\tilde{f}} = \tilde{f}$ a.e. in $(-T,T)$ and $\tilde{\tilde{f}}(c_0) = \tilde{f}(c_0)$. Thus,

$$\tilde{\tilde{f}}(d) := \tilde{\tilde{f}}(c_0) + \int_{c_0}^d (-\tilde{B}\tilde{\tilde{f}})(x)dx \quad \text{for all } d \in (-T,T).$$

If we take $d_1, d_2 \in (-T,T)$, then we have

$$\tilde{\tilde{f}}(d_1) = \tilde{\tilde{f}}(c_0) + \int_{c_0}^{d_1} (-\tilde{B}\tilde{\tilde{f}})(x)dx$$

and

$$\tilde{\tilde{f}}(d_2) = \tilde{\tilde{f}}(c_0) + \int_{c_0}^{d_2} (-\tilde{B}\tilde{\tilde{f}})(x)dx.$$

Hence, we get

$$\tilde{\tilde{\tilde{f}}}(d_2)=\tilde{\tilde{\tilde{f}}}(d_1)+\int_{d_1}^{d_2}(-\tilde{B}\tilde{\tilde{\tilde{f}}})(x)dx \quad \text{for all } d_1,d_2\in(-T,T).$$

Thus, according to the Theorem 28, $\tilde{\tilde{\tilde{f}}}$ is an absolutely continuous function with derivative (almost everywhere) equal to $-\tilde{B}\tilde{\tilde{\tilde{f}}}\in\widetilde{L}^2(-T,T)$. Thus, we have

$$D(\tilde{B})\subset D(\tilde{A}) \quad \text{and} \quad \tilde{A}\big|_{D(\tilde{B})}=\tilde{B}. \tag{72}$$

2. In the second step, we will deduce that $\tilde{B}=\tilde{A}$. To this purpose, we make some observations:
(i) The semigroup $(\tilde{G}_r(t))_{t\geq 0}$ is a contractive semigroup, so there is

$$\|\tilde{G}_r(t)\|_{L^2(-T,T)}\leq 1$$

for all $t\geq 0$. Hence, from Theorem 12 we obtain that $1\in\rho(\tilde{B})$.
(ii) We will also show below that $1\in\rho(\tilde{A})$. We know that

$$1\in\rho(\tilde{A})\Leftrightarrow \text{exists the bounded operator } (\tilde{A}-I)^{-1} \text{ on } \widetilde{L}^2(-T,T).$$

We can see that for $\tilde{f}\in\widetilde{L}^2(-T,T)$

$$(\tilde{A}-I)^{-1}\tilde{f}=\tilde{u}\Leftrightarrow\tilde{f}=(\tilde{A}-I)\tilde{u}\Leftrightarrow\tilde{f}=-\tilde{u}'-\tilde{u},$$

where $\tilde{u}\in D(\tilde{A})$. Thus, $1\in\rho(\tilde{A})$ if and only if for all $\tilde{f}\in\widetilde{L}^2(-T,T)$ there exists a unique solution of the following equation:

$$\tilde{f}=-\tilde{u}'-\tilde{u}, \tag{73}$$

and this solution belongs to $D(\tilde{A})$. It is easy to see that the solution of (73) is given by

$$\tilde{u}(t)=-\int_{-T}^{t}e^{s-t}\tilde{f}(s)ds.$$

Thus, we have

$$((\tilde{A}-I)^{-1}\tilde{f})(t)=-\int_{-T}^{t}e^{s-t}\tilde{f}(s)ds=-\int_{-T}^{t}e^{-(t-s)}\tilde{f}(s)ds=-(\tilde{f}*e^{-s})(t),$$

and then from Young's convolution inequality we get

$$\begin{aligned}\|(\tilde{A}-I)^{-1}\tilde{f}\|_{L^2(-T,T)}&=\|\tilde{f}*e^{-s}\|_{L^2(-T,T)}\leq\|e^{-s}\|_{L^1(-T,T)}\|\tilde{f}\|_{L^2(-T,T)}\\&=|e^{-b}-e^{-a}|\|\tilde{f}\|_{L^2(-T,T)}.\end{aligned}$$

Hence, $1\in\rho(\tilde{A})$.
(iii) Due to (72) and observation (i), we obtain

$$(I-\tilde{A})(D(\tilde{B}))=(I-\tilde{B})(D(\tilde{B}))=\widetilde{L}^2(-T,T).$$

Moreover, using observation (ii), we have

$$D(\tilde{A}) = (I - \tilde{A})^{-1}(\widetilde{L}^2(-T,T)).$$

Hence, we get

$$D(\tilde{A}) = (I - \tilde{A})^{-1}(I - \tilde{A})(D(\tilde{B})) = D(\tilde{B}),$$

and then $\tilde{A} = \tilde{B}$.

$\square$

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# A SELF-SIMILAR SOLUTION TO SPACE-FRACTIONAL STEFAN PROBLEM

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**Abstract:** In this paper, we find a self-similar solution to the one-phase, one-dimensional, space-fractional Stefan problem. We assume that the non-local flux is given in terms of space-fractional Caputo derivative of transported substance.
**Keywords:** fractional Stefan problem, Self-similar solutions, Caputo derivative
**Mathematics Subject Classification (2020):** 35C06 (primary), 35R11, 35R37

## 1. INTRODUCTION

In this paper, we find a special solution to the space-fractional, one-phase, one-dimensional Stefan problem

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^{\alpha} u = 0 & \text{in } \{(x,t) : 0 < x < s(t),\ \ 0 < t < \infty\}, \\ u(0,t) = c_1,\ \ u(t,s(t)) = 0 & \text{for } t \in (0,\infty), \\ \dot{s}(t) = -(D^{\alpha}u)(s(t),t) & \text{for } t \in (0,\infty), \end{cases} \tag{1}$$

where we assume that $\alpha \in (0,1)$, $s(0) = 0$ and $c_1 > 0$. This is a non-linear problem with a pair of unknowns $(u,s)$, where $u$ may be regarded as temperature of a medium or density of a transported substance, while $s : \mathbb{R}_+ \to \mathbb{R}$ denotes a moving boundary of the domain. We consider one-phase problem, hence we assume that $u(x,t) = 0$ for $x > s(t)$.

By $D^{\alpha}$ we denote the fractional Caputo derivative with respect to the spatial variable given by

$$D^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} u_x(p,t)\,dp. \tag{2}$$

The problem (1) may be derived from the balance law, assuming the following non-local form of the flux:

$$q(x,t) = \begin{cases} -D^{\alpha}u(x,t) & \text{in } \{(x,t) : 0 < x < s(t),\ 0 < t < \infty\} \\ 0 & \text{in } \{(x,t) : s(t) < x < \infty,\ 0 < t < \infty\}. \end{cases} \tag{3}$$

The idea of representing the diffusive flux in terms of space-fractional Caputo derivative was proposed in paper [5], where the model of infiltration of water into heterogeneous soils was considered. Motivation to study problem (1) comes from [6], where the author considered one-phase, one-dimensional Stefan problem with diffusive flux given by (3). It is worth mentioning that the existence of a unique classical solution to the problem

$$\begin{cases} u_t - \frac{\partial}{\partial x}D^{\alpha}u = 0 & \text{in } \{(x,t) : 0 < x < s(t),\ 0 < t < T\}, \\ u_x(0,t) = 0,\ \ u(t,s(t)) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0 & \text{on } (0,b), \\ \dot{s}(t) = -(D^{\alpha}u)(s(t),t) & \text{for } t \in (0,T), \end{cases} \tag{4}$$

where $T, b > 0$, $s(0) = b$, was proved in [3]. The aim of this article is to obtain the exact formula for a solution in the case when $b = 0$ and with a positive, constant Dirichlet condition on the left boundary, i.e. problem (1). The results of this article come from the author's PhD Thesis. It must also be mentioned that the self-similar solution to space-fractional Stefan problem was obtained independently in the recent paper [2]. However, here we give an independent proof of this result.

Before we proceed to the construction of the self-similar solution, let us recall some preliminary facts from fractional-calculus, which will be used in the paper. We begin with the definition of the fractional integral and the Riemann-Liouville fractional derivative.

**Definition 1.** *Let $L > 0$, $\alpha > 0$. For $f \in L^1(0,L)$ we introduce the fractional integral $I^{\alpha}$ by the formula*

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-p)^{\alpha-1}f(p)dp. \tag{5}$$

*If $\alpha \in (0,1)$ and $f$ is regular enough (for example, $f$ is absolutely continuous), we may define the Riemann-Liouville fractional derivative as*

$$\partial^{\alpha}f(x) = \frac{\partial}{\partial x}I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}f(p)dp.$$

*Here, $\Gamma(\cdot)$ denotes the Gamma function given by the formula*

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt.$$

It follows from (2) that $D^{\alpha}f = I^{1-\alpha}f'$. Thus, we may represent the diffusive operator $\frac{\partial}{\partial x}D^{\alpha}$ equivalently as

$$\frac{\partial}{\partial x}D^{\alpha}u = \frac{\partial}{\partial x}I^{1-\alpha}u_x = \partial^{\alpha}u_x. \tag{6}$$

Let us now recall the superposition property for fractional integrals.

**Proposition 2.** *[4, Theorem 2.5.] Let $\alpha, \beta, L > 0$, $f \in L^1(0,L)$. Then,*

$$I^{\alpha} I^{\beta} f = I^{\alpha+\beta} f.$$

One of the fundamental issues in solving the fractional differential equations with the Caputo derivative is to investigate whether the operator $I^{\alpha}$ acts like an operator inverse to $D^{\alpha}$. Here we cite Lemma 2.21 from [1], however, instead of $L^{\infty}$ we assume $L^p$ regularity.

**Proposition 3.** *[1, Lemma 2.21] Let $L > 0$, $\alpha \in (0,1)$. Then, we have*

$$(D^{\alpha} I^{\alpha} f)(x) = f(x) \quad for \quad f \in L^p(0,L), \;\; p > \frac{1}{\alpha},$$

$$(I^{\alpha} D^{\alpha} f)(x) = f(x) - f(0) \quad for \quad f \in AC[0,L].$$

Now, we present an analogous result in the case of the fractional Riemann-Liouville derivative.

**Proposition 4.** *[4, Theorem 2.4] Let $\alpha \in (0,1)$ and $L > 0$. Then,*

$$\partial^{\alpha} I^{\alpha} f = f \quad for \quad f \in L^1(0,L).$$

*If $f \in L^1(0,L)$ is such that $\partial^{\alpha} f \in L^1(0,L)$, we have*

$$I^{\alpha} \partial^{\alpha} f(x) = f(x) - \frac{x^{\alpha-1}}{\Gamma(\alpha)} I^{1-\alpha} f(0), \quad where \quad I^{1-\alpha} f(0) := \lim_{x \to 0} I^{1-\alpha} f(x).$$

*We note that the limit is well defined, because by the assumption $I^{1-\alpha} f$ is absolutely continuous. In particular, if $f$ additionally belongs to $L^p(0,L)$ for $p > \frac{1}{1-\alpha}$, then*

$$I^{\alpha} \partial^{\alpha} f = f.$$

Now, we present how the fractional operators act on polynomial functions.

**Proposition 5.** *Let $\alpha \in (0,1)$, $\beta > -1$. Then,*

$$I^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}$$

*and for $\beta > 0$*

$$\partial^{\alpha} x^{\beta} = D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}.$$

We finish this section with a result concerning a sign of the operator $\frac{\partial}{\partial x} D^{\alpha} f$ in the extremum point of $f$. Here, we replace the regularity assumptions which appear in [3, Lemma 7] with more natural ones. Next, we will make use of this lemma in order to show that our self-similar solution is non-negative.

**Lemma 6.** *Let $\alpha \in (0,1)$, $f : [0,L] \to \mathbb{R}$ be an absolutely continuous function such that there exists $\gamma > \frac{1}{2}$ such that $f' \in H^{\alpha+\gamma}(\varepsilon, L)$ for every $\varepsilon > 0$. Then, $\frac{\partial}{\partial x} D^\alpha f$ is continuous on $(0,L]$ and*

1. *if $f$ attains its local maximum at $x_0 \in (0,L)$ which is a global maximum on $[0,x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) \leq 0$. Furthermore, if $f$ is not constant on $[0,x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) < 0$;*
2. *if $f$ attains its local minimum at $x_0 \in (0,L)$ which is a global minimum on $[0,x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) \geq 0$. Furthermore, if $f$ is not constant on $[0,x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) > 0$.*

*Proof.* Let us begin with the proof of continuity of $\frac{\partial}{\partial x} D^\alpha f$. To this end, we take $x_1, x \in (0,L)$. Let us assume that $x_1 < x$. The case $x < x_1$ can be shown analogously. We note that for every $0 < \varepsilon < y < L$,

$$\begin{aligned}\Gamma(1-\alpha)(\frac{\partial}{\partial x} D^\alpha f)(y) &= \frac{\partial}{\partial y}\int_0^\varepsilon (y-p)^{-\alpha} f'(p)dp + \frac{\partial}{\partial y}\int_\varepsilon^y (y-p)^{-\alpha} f'(p)dp \\ &= -\alpha \int_0^\varepsilon (y-p)^{-\alpha-1} f'(p)dp + \frac{\partial}{\partial y}\int_\varepsilon^y (y-p)^{-\alpha}[f'(p) - f'(\varepsilon)]dp \\ &\quad + f'(\varepsilon)(y-\varepsilon)^{-\alpha}.\end{aligned}$$

Let us denote $\partial_\varepsilon^\alpha g(x) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_\varepsilon^x (x-p)^{-\alpha} g(p)dp$. Then, taking arbitrary $\varepsilon \in (0,x_1)$, we obtain

$$\begin{aligned}\Gamma(1-\alpha)\left|\frac{\partial}{\partial x} D^\alpha f(x) - \frac{\partial}{\partial x} D^\alpha f(x_1)\right| &\leq \alpha \int_0^\varepsilon [(x_1-p)^{-\alpha-1} - (x-p)^{-\alpha-1}] \left|f'(p)\right| dp \\ &\quad + \Gamma(1-\alpha)\left|\partial_\varepsilon^\alpha [f' - f'(\varepsilon)](x) - \partial_\varepsilon^\alpha [f' - f'(\varepsilon)](x_1)\right| \\ &\quad + \left|f'(\varepsilon)\right| [(x_1-\varepsilon)^{-\alpha} - (x-\varepsilon)^{-\alpha}].\end{aligned}$$

The first term tends to zero as $x \to x_1$, because convergence under the integral is uniform. By [3, Corollary 1], we have

$$I_\varepsilon^{1-\alpha}[f' - f'(\varepsilon)] = \frac{1}{\Gamma(1-\alpha)} \int_\varepsilon^x (x-p)^{-\alpha}[f' - f'(\varepsilon)]dp \in {}_0H^{1+\gamma}(\varepsilon, L),$$

where the space ${}_0H^{1+\gamma}$ is defined in [3, Corollary 1]. Hence, we obtain that $\partial_\varepsilon^\alpha [f' - f'(\varepsilon)] \in H^\gamma(\varepsilon, L) \hookrightarrow C[\varepsilon, L]$. Thus, continuity of $\frac{\partial}{\partial x} D^\alpha f$ on $(0,L)$ is proved. We will prove only the part of the claim concerning maximum, because the proof of the second part is analogous. We define $g(x) = f(x_0) - f(x)$. Then $g$ is non negative on $[0,x_0]$, $g'(x_0) = 0$ and $\frac{\partial}{\partial x} D^\alpha g = -\frac{\partial}{\partial x} D^\alpha f$. We note that by the Sobolev embedding $g \in C^{0,\beta}[\varepsilon, L]$ for $\beta = \alpha + \gamma - \frac{1}{2} > \alpha$ and for every $\varepsilon > 0$. Hence, for every $0 < \varepsilon < x \leq x_0$ we may estimate as follows:

$$\left|g'(x)\right| = \left|g'(x) - g'(x_0)\right| \leq c\,|x - x_0|^\beta \tag{7}$$

and

$$g(x) \leq \int_x^{x_0} \left|g'(p)\right| dp \leq \frac{c}{\beta+1} |x - x_0|^{\beta+1}. \tag{8}$$

Using these estimates we may differentiate under the integral sign as follows

$$
\begin{aligned}
(\frac{\partial}{\partial x} D^{\alpha} g)(x_0) &= \frac{1}{\Gamma(1-\alpha)} \left( \frac{\partial}{\partial x} \int_0^x (x-p)^{-\alpha} g'(p) dp \right)(x_0) \\
&= \frac{1}{\Gamma(1-\alpha)} \lim_{p \to x_0^-} (x_0 - p)^{-\alpha} g'(p) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-1} g'(p) dp
\end{aligned}
$$

and the limit is equal to zero by the estimate (7). Integrating by parts we obtain further

$$
\begin{aligned}
(\frac{\partial}{\partial x} D^{\alpha} g)(x_0) = -\frac{\alpha}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-1} g'(p) dp = -\frac{\alpha}{\Gamma(1-\alpha)} \lim_{p \to x_0^-} (x_0 - p)^{-\alpha-1} g(p) \\
+ \frac{\alpha}{\Gamma(1-\alpha)} x_0^{-\alpha-1} g(0) + \frac{\alpha(\alpha+1)}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-2} g(p) dp.
\end{aligned}
$$

By (8), the limit equals zero, hence we arrive at

$$
(\frac{\partial}{\partial x} D^{\alpha} g)(x_0) = \frac{\alpha}{\Gamma(1-\alpha)} x_0^{-\alpha-1} g(0) + \frac{\alpha(\alpha+1)}{\Gamma(1-\alpha)} \int_0^{x_0} (x_0 - p)^{-\alpha-2} g(p) dp
$$

and

$$
(\frac{\partial}{\partial x} D^{\alpha} g)(x_0) \geq 0, \text{ which implies } (\frac{\partial}{\partial x} D^{\alpha} f)(x_0) \leq 0.
$$

Furthermore, from the formula above we obtain that if $f$ is not a constant function on $[0, x_0]$, then $(\frac{\partial}{\partial x} D^{\alpha} f)(x_0) < 0$. □

## 2. SIMILARITY VARIABLE

We would like to find a scale-invariant solution to problem (1). In order to find the appropriate scaling, we introduce

$$
u^{\lambda}(x,t) := \lambda^c u(\lambda^a x, \lambda^b t) \quad \text{for} \quad a,b,c,\lambda > 0.
$$

Let us perform the calculations

$$
u_t(\lambda^a x, \lambda^b t) = \lambda^{-b-c} u_t^{\lambda}(x,t) \text{ and } u_x(\lambda^a x, \lambda^b t) = \lambda^{-a-c} u_x^{\lambda}(x,t).
$$

Next, we have

$$
\Gamma(1-\alpha) \frac{\partial}{\partial x} D^{\alpha} u^{\lambda}(x,t) = \frac{\partial}{\partial x} \int_0^x (x-p)^{-\alpha} u_x^{\lambda}(p,t) dp = \frac{\partial}{\partial x} \int_0^x (x-p)^{-\alpha} \lambda^{a+c} u_x(\lambda^a p, \lambda^b t) dp.
$$

Substituting $\lambda^a p = w$, we obtain

$$
\Gamma(1-\alpha) \partial^{\alpha} u_x^{\lambda}(x,t) = \lambda^c \frac{\partial}{\partial x} \int_0^{\lambda^a x} (x - w\lambda^{-a})^{-\alpha} u_x(w, \lambda^b t) dw
$$

$$= \lambda^{a\alpha+c}\frac{\partial}{\partial x}\int_0^{\lambda^a x}(\lambda^a x - w)^{-\alpha}u_x(w,\lambda^b t)dw$$
$$= \lambda^{a(\alpha+1)+c}(\frac{\partial}{\partial x}D^\alpha u)(\lambda^a x, \lambda^b t).$$

Hence, if $u$ satisfies $(1)_1$, then

$$0 = \lambda^{-c}\lambda^{-b}u_t^\lambda(x,t) - \lambda^{-c}\lambda^{-a(\alpha+1)}\frac{\partial}{\partial x}D^\alpha u^\lambda(x,t).$$

We are looking for a self-similar solution, so if we take $c = 0$ and suppose that $u \equiv u^\lambda$, we arrive at

$$b = a(\alpha+1).$$

Motivated by the above calculation, we introduce the similarity variable $\xi = xt^{-\frac{1}{\alpha+1}}$ and we define

$$F(\xi) = F(xt^{-\frac{1}{\alpha+1}}) := u(x,t).$$

Let us rewrite the equation $(1)_1$ in terms of function $F$. Then

$$u_t(x,t) = -\frac{1}{\alpha+1}xt^{-\frac{1}{\alpha+1}-1}F'(\xi), \;\; u_x(x,t) = t^{-\frac{1}{\alpha+1}}F'(\xi) \tag{9}$$

and

$$\Gamma(1-\alpha)\frac{\partial}{\partial x}D^\alpha u(x,t) = t^{-\frac{1}{\alpha+1}}\frac{\partial}{\partial x}\int_0^x (x-p)^{-\alpha}F'(pt^{-\frac{1}{\alpha+1}})dp$$
$$= \frac{\partial}{\partial x}\int_0^{xt^{-\frac{1}{1+\alpha}}}(x - wt^{\frac{1}{\alpha+1}})^{-\alpha}F'(w)dw$$
$$= t^{-\frac{\alpha}{\alpha+1}}\frac{\partial}{\partial x}\int_0^{xt^{-\frac{1}{\alpha+1}}}(xt^{-\frac{1}{\alpha+1}} - w)^{-\alpha}F'(w)dw$$
$$= \Gamma(1-\alpha)t^{-1}\frac{\partial}{\partial \xi}D^\alpha F(\xi). \tag{10}$$

Hence, if $u$ satisfies $(1)_1$, recalling the identity (6) we obtain

$$-\frac{1}{1+\alpha}\xi F'(\xi) - \partial^\alpha F'(\xi) = 0.$$

## 3. A SELF-SIMILAR SOLUTION

In this section, we will proceed as follows: at first, we will solve the auxiliary problem for function $F$ with boundary conditions $F(0) = c_1$, $I^{1-\alpha}F'(0) = c_2$ on the interval $[0,R]$, where $R>0$, $c_2<0$ are arbitrary constants and $c_1$ comes from $(1)_2$. Then, we will propose the formula for the family $\{s^R\}_{R>0}$ and we will choose the constant $c_2=c_2(R)$ such that the pair $u^R(x,t)=F^R(xt^{-\frac{1}{1+\alpha}})$ and $s^R$ is a solution to $(1)_1$, $(1)_3$ . Finally, we will choose $R=c_0>0$ such that $F(c_0)=0$, which will guarantee that the pair $(u^{c_0},s^{c_0})$ satisfies the whole system (1).

**Lemma 7.** *Let us consider the problem*

$$\begin{cases} \partial^\alpha F'(\xi)=-\frac{\xi}{\alpha+1}F'(\xi) & \text{for} \quad 0<\xi<R, \\ F(0)=c_1, \quad I^{1-\alpha}F'(0)=c_2, \end{cases} \tag{11}$$

*where $c_1>0$, $R>0$, $c_2<0$ are fixed constants and $I^{1-\alpha}F'(0):=\lim_{\xi\to 0}I^{1-\alpha}F'(\xi)$. Then, there exists exactly one solution to (11) which belongs to*

$$X_{R,c_1,c_2}:=\{v\in C^1((0,R]):\xi^{1-\alpha}v'\in C([0,R]),\ v(0)=c_1,\ I^{1-\alpha}v'(0)=c_2\}.$$

*Furthermore, the solution is given by the formula*

$$F(\xi)=c_1+\frac{c_2}{\Gamma(\alpha+1)}\left[\xi^\alpha+\Gamma(\alpha+1)\xi^\alpha\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k\frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right], \tag{12}$$

*where the series is uniformly convergent on $[0,R]$. Finally, if we define*

$$u(x,t):=F(xt^{-\frac{1}{1+\alpha}}), \tag{13}$$

*then $u(0,t)=c_1$ and $u$ satisfies $(1)_1$ on $\{(x,t):0<x<Rt^{\frac{1}{\alpha+1}},0<t<\infty\}$.*

*Proof.* At first, we will rewrite (11) in the integral form. Let us assume that $F$ belonging to $X_{R,c_1,c_2}$ satisfies (11). We apply $I^\alpha$ to both sides of $(11)_1$. Since $F'\in L^1(0,R)$, from identity (11) we obtain that $\partial^\alpha F'\in L^1(0,R)$ as well. Hence, we may apply Proposition 4 to obtain

$$F'(\xi)=c_2\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}-\frac{1}{\alpha+1}I^\alpha(\xi F')(\xi). \tag{14}$$

Integrating this identity and applying Proposition 2, we arrive at

$$F(\xi)=c_1+\frac{\xi^\alpha}{\Gamma(\alpha+1)}c_2-\frac{1}{\alpha+1}I^\alpha I(\xi F')(\xi).$$

We note that

$$\int_0^\xi pF'(p)dp=\xi F(\xi)-\int_0^\xi F(p)dp, \quad \text{i.e.} \quad I(\xi F')=\xi F-IF.$$

Denoting by $E$ the identity operator, we get

$$F(\xi)=c_1+\frac{\xi^\alpha}{\Gamma(\alpha+1)}c_2+\frac{1}{\alpha+1}I^\alpha(I-\xi E)F(\xi).$$

The above identity may be written in the following form:

$$F(\xi) = G(\xi) + KF(\xi), \tag{15}$$

where

$$G(\xi) = c_1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} c_2, \quad KF(\xi) = \frac{1}{\alpha+1} I^\alpha (I - \xi E) F(\xi).$$

Let us find a solution to (15). Applying the operator $K$ to both sides of (15), we obtain

$$KF(\xi) = KG(\xi) + K^2 F(\xi),$$

and thus

$$F(\xi) = G(\xi) + KG(\xi) + K^2 F(\xi).$$

Applying subsequent powers of $K$ to (15), we arrive at

$$F(\xi) = \sum_{k=0}^{n} K^k G(\xi) + K^{n+1} F(\xi) \quad \text{for any} \quad n \in \mathbb{N}. \tag{16}$$

We note that if $F$ belongs to $C([0,R])$, then $K^n F \to 0$ uniformly on $[0,R]$. Indeed, using Proposition 5, we may calculate that

$$I^\alpha (I + \xi E) \xi^\beta = \frac{\Gamma(\beta+3)}{\Gamma(\beta+\alpha+2)(\beta+1)} \xi^{\beta+\alpha+1}.$$

Hence, we have

$$\begin{aligned} |K^n F(\xi)| &\le \|F\|_{C([0,R])} \frac{1}{(\alpha+1)^n} (I^\alpha (I + \xi E))^n 1 = \|F\|_{C([0,R])} \frac{\xi^{n(\alpha+1)} \prod_{k=0}^{n-1} (k(\alpha+1)+3)}{(1+\alpha)^n \Gamma(n(\alpha+1)+1)} \\ &= \|F\|_{C([0,R])} \frac{\xi^{n(\alpha+1)} \prod_{k=0}^{n-1} (k + \frac{3}{\alpha+1})}{\Gamma(n(\alpha+1)+1)} \le 2 \|F\|_{C([0,R])} R^{n(\alpha+1)} \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha n)} \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus, we may pass to the limit in (16) to obtain

$$F(\xi) = \sum_{k=0}^{\infty} K^k G(\xi). \tag{17}$$

We will show that the series is uniformly convergent on $[0,R]$ and we will find its sum. First, we note that for any $n \in \mathbb{N} \setminus \{0\}$ we have $(I - \xi E)^n 1 = 0$. Thus

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{1}{(1+\alpha)^k} [I^\alpha (I - \xi E)]^k \xi^\alpha.$$

Furthermore, from Proposition 5 we infer that

$$I^\alpha (I - \xi E) \xi^\beta = -\frac{\beta \Gamma(\beta+1)}{\Gamma(\beta+\alpha+2)} \xi^{\beta+\alpha+1}. \tag{18}$$

We will show by induction that for every $k \in \mathbb{N}$, $k \geq 1$ we have

$$\frac{1}{\Gamma(1+\alpha)}[I^{\alpha}(I-\xi E)]^{k}\xi^{\alpha} = (-\xi^{1+\alpha})^{k}\xi^{\alpha}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}. \tag{19}$$

For $k=1$, applying (18) with $\beta=\alpha$, we arrive at

$$\frac{1}{\Gamma(1+\alpha)}I^{\alpha}(I-\xi E)\xi^{\alpha} = -\xi^{2\alpha+1}\frac{\alpha}{\Gamma(2\alpha+2)},$$

which is (19) with $k=1$. Let us assume that for a fixed $k \geq 1$ identity (19) is satisfied. Then

$$\frac{1}{\Gamma(1+\alpha)}[I^{\alpha}(I-\xi E)]^{k+1}\xi^{\alpha} = \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}I^{\alpha}(I-\xi E)[(-\xi^{1+\alpha})^{k}\xi^{\alpha}].$$

Using (18) with $\beta=(1+\alpha)k+\alpha$, we get

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}[I^{\alpha}(I-\xi E)]^{k+1}\xi^{\alpha} &= \prod_{i=1}^{k}(i\alpha+i-1)\cdot\frac{(-1)^{k+1}[(1+\alpha)k+\alpha]}{\Gamma((1+\alpha)k+2\alpha+2)}\xi^{(1+\alpha)k+2\alpha+1}\\ &= (-\xi^{1+\alpha})^{k+1}\xi^{\alpha}\frac{\prod_{i=1}^{k+1}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+2))}.\end{aligned}$$

Hence, by the principle of mathematical induction, we obtain (19). From (19), it follows that the function $F$ defined by (17) is given by the formula

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)}\left[\xi^{\alpha}+\Gamma(\alpha+1)\xi^{\alpha}\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^{k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right].$$

We will show that the series above is uniformly absolutely convergent. Indeed, let us denote

$$a_k = \frac{R^{(1+\alpha)k+\alpha}}{(1+\alpha)^{k}}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}.$$

Then,

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= R^{\alpha+1}\frac{k(\alpha+1)+\alpha}{1+\alpha}\frac{\Gamma((\alpha+1)k+\alpha+1)}{\Gamma((\alpha+1)k+2(\alpha+1))}\\ &\leq \frac{R^{\alpha+1}}{\alpha+1}\frac{\Gamma((\alpha+1)k+\alpha+2)}{\Gamma((\alpha+1)k+\alpha+2+\alpha)} = \frac{R^{\alpha+1}}{\alpha+1}\frac{B(\alpha,(\alpha+1)k+\alpha+2)}{\Gamma(\alpha)} \longrightarrow 0 \text{ as } k\to\infty.\end{aligned}$$

Thus, by the Weierstrass and d'Alembert criteria the series in (12) is uniformly absolutely convergent. Now we will check whether $F$ defined by (12) actually satisfies (15). Let us calculate $KF$. We note that

$$\frac{1}{\alpha+1}I^{\alpha}(I-\xi E)c_1 = 0,$$

hence

$$KF(\xi) = \frac{1}{\alpha+1}I^{\alpha}(I-\xi E)\left[\frac{c_2}{\Gamma(\alpha+1)}\left[\xi^{\alpha}+\Gamma(\alpha+1)\xi^{\alpha}\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^{k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right]\right].$$

Integrating the series term by term and using (18), we obtain

$$\begin{aligned}
&\frac{1}{\alpha+1} I^{\alpha}(I-\xi E)F(\xi)\\
&\quad = -\frac{\alpha}{\alpha+1} c_2 \frac{\xi^{2\alpha+1}}{\Gamma(2(\alpha+1))}\\
&\qquad - \frac{1}{(\alpha+1)} c_2 \sum_{k=1}^{\infty} \left(\frac{-1}{1+\alpha}\right)^k \frac{\xi^{(1+\alpha)k+2\alpha+1}}{\Gamma((1+\alpha)(k+2))} [(1+\alpha)k+\alpha] \prod_{i=1}^{k} (i\alpha+i-1)\\
&\quad = -\frac{\alpha}{\alpha+1} c_2 \frac{\xi^{2\alpha+1}}{\Gamma(2(\alpha+1))} + c_2 \xi^{\alpha} \sum_{k=2}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((1+\alpha)(k+1))}\\
&\quad = c_2 \xi^{\alpha} \sum_{k=1}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((1+\alpha)(k+1))}.
\end{aligned}$$

Hence, we have verified that the function $F$ given by (12) satisfies (15). Furthermore, the solution to (15) belongs to $X_{R,c_1,c_2}$. Indeed, $F$ given by (12) is continuous as a uniform limit of a sequence of continuous functions. By (15), we obtain $F(0) = c_1$ and

$$(I^{1-\alpha}F')(0) = c_2 + \frac{1}{1+\alpha}(D^{\alpha}I^{\alpha}(I-\xi E)F)(0) = c_2 + \frac{1}{1+\alpha}((I-\xi E)F)(0) = c_2.$$

In order to show $\xi^{1-\alpha}F' \in C([0,R])$, we differentiate the series in (12) term by term.

$$\frac{d}{d\xi}\left[\xi^{\alpha} \sum_{k=1}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right] = \xi^{\alpha-1} \sum_{k=1}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1)+1)}. \tag{20}$$

We will show that this series is absolutely uniformly convergent on $[0,R]$. Let us denote

$$b_k = \left(\frac{R^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1)+1)}.$$

Then,

$$\begin{aligned}
\frac{b_{k+1}}{b_k} &= \frac{R^{1+\alpha}}{1+\alpha} \cdot \frac{[(k+1)(\alpha+1)-1]\Gamma((\alpha+1)(k+1)+1)}{\Gamma((\alpha+1)(k+2)+1)}\\
&\leq \frac{R^{1+\alpha}}{1+\alpha} \cdot \frac{\Gamma((\alpha+1)(k+1)+2)}{\Gamma((\alpha+1)(k+2)+1)} = \frac{R^{1+\alpha}B(\alpha,(\alpha+1)(k+1)+2)}{(1+\alpha)\Gamma(\alpha)} \longrightarrow 0, \text{ as } k \to \infty.
\end{aligned}$$

Hence, the series in (20) is uniformly absolutely convergent, which leads to $\xi^{1-\alpha}F' \in C([0,R])$. Now we will show that $F$ satisfies (11). Since $F' \in L^1(0,R)$, we may apply $D^{\alpha}$ to both sides of (15) to obtain

$$D^{\alpha}F(\xi) = c_2 + \frac{1}{1+\alpha} IF(\xi) - \frac{\xi}{1+\alpha} F(\xi),$$

where we made use of Proposition 3 and Proposition 5. The right-hand-side is absolutely continuous, hence by differentiating the identity above we arrive at

$$\frac{\partial}{\partial x} D^\alpha F(\xi) = -\frac{\xi}{1+\alpha} F'(\xi).$$

The identities (9) and (10) finish the proof. □

**Lemma 8.** *Let $F$ be a solution to the problem (11) given in Lemma 7. Then, for every $R>0$ there holds $F'<0$ on $(0,R)$. Furthermore, function $u$ defined by (13) satisfies $u_t>0$, $u_x<0$ on $\{(x,t): 0<x<Rt^{\frac{1}{\alpha+1}}, 0<t<\infty\}$.*

*Proof.* Since $c_2<0$, by (12) we have

$$F'(\xi) \to -\infty \quad \text{as} \quad \xi \to 0.$$

Indeed, the derivative of the series in (12) vanishes as $\xi \to 0$ and $c_2\xi^{\alpha-1} \to -\infty$ as $\xi \to 0$. Hence, $F$ is decreasing in the neighborhood of zero. We note that $F$ satisfies the assumptions of Lemma 6, because by Lemma 7 it is absolutely continuous and smooth away from the origin. Let us assume that $F$ admits a local minimum at point $\xi_0>0$. Then, $F'(\xi_0)=0$ and, since $F$ is not constant, by Lemma 6 we obtain $(\frac{\partial}{\partial x} D^\alpha F)(\xi_0)<0$. It leads to a contradiction with (11). Thus, $F'<0$. The final part follows from (9). □

In the next lemma, we obtain the family $(u^R, s^R)_{R>0}$ of solutions to $(1)_1$ and $(1)_3$.

**Lemma 9.** *For every $c_1>0$ and every $R>0$, the functions*

$$s^R(t) = Rt^{\frac{1}{1+\alpha}}, \tag{21}$$

$$u^R(x,t) = c_1 + \frac{\tilde{c}_2}{\Gamma(\alpha+1)}\left[x^\alpha t^{-\frac{\alpha}{\alpha+1}} + \Gamma(\alpha+1)x^\alpha t^{-\frac{\alpha}{\alpha+1}} \sum_{k=1}^\infty \left(\frac{-x^{1+\alpha}}{(1+\alpha)t}\right)^k \frac{\prod_{i=1}^k (i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right] \tag{22}$$

*where*

$$\tilde{c}_2 = -\frac{R}{(1+\alpha)\left[1+\sum_{k=1}^\infty \left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k (i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}\right]} \tag{23}$$

*satisfy the equation $(1)_3$. Moreover, $u^R$ is a solution to $(1)_1$ with $s(t)=s^R(t)$ and $u^R(0,t)=c_1$.*

*Proof.* We note that $u^R(x,t) = F(xt^{-\frac{1}{1+\alpha}})$ where $F$ is the solution to (11) with $c_2$ equal to $\tilde{c}_2$ whenever $\tilde{c}_2$ given by (23) is well defined and negative. It is enough to show that the denominator in the definition of $\tilde{c}_2$ is positive. To this end, let us recall the formula for the function $F$ given by (12). By Lemma 8, for any $c_2<0$ there holds $F'<0$. Thus, we have also $D^\alpha F<0$. Applying Proposition 5, we deduce that for any $c_2<0$

$$D^\alpha F(R) = c_2\left[1+\sum_{k=1}^\infty \left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k (i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}\right].$$

This implies that

$$1+\sum_{k=1}^{\infty}\left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^{k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}>0.$$

Hence, for every $R>0$ the constant $\tilde{c}_2$ given by (23) is well defined and negative. By Lemma 7, the function $u^R$ fulfills $(1)_1$ with $s(t)=s^R(t)$ and $u^R(0,t) = c_1$. Moreover,

$$t^{-\frac{\alpha}{\alpha+1}}I^{1-\alpha}F'(\xi)=(I^{1-\alpha}u_x^R)(x,t),$$

hence, $I^{1-\alpha}F'(0)=\tilde{c}_2$ implies $(I^{1-\alpha}u_x^R)(0,t)=\tilde{c}_2t^{-\frac{\alpha}{\alpha+1}}$. Now we will show that $(u^R,s^R)_{R>0}$ given by (21) - (23) satisfy $(1)_3$. Let us calculate $D^\alpha u^R(x,t)$ for $u^R$ given by (22). From Proposition 5, we get

$$D^\alpha u^R(x,t)=\tilde{c}_2t^{-\frac{\alpha}{\alpha+1}}+t^{-\frac{\alpha}{\alpha+1}}\tilde{c}_2\sum_{k=1}^{\infty}\left(\frac{-x^{1+\alpha}}{(1+\alpha)t}\right)^{k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}.$$

Hence, for $s^R$ given by (21) we have

$$t^{\frac{\alpha}{\alpha+1}}D^\alpha u^R(s^R(t),t)=\tilde{c}_2+\tilde{c}_2\sum_{k=1}^{\infty}\left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^{k}\frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}.$$

Applying (23), we obtain

$$-D^\alpha u^R(s^R(t),t)=t^{-\frac{\alpha}{\alpha+1}}\frac{R}{1+\alpha}=\frac{d}{dt}s^R(t).$$

Hence, the functions $s^R$ and $u^R$ defined by (21) and (22) satisfy $(1)_3$, which completes the proof. □

It remains to choose $R>0$ such that the pair $(u^R,s^R)$ given by Lemma (9) satisfies $u^R(s^R(t),t)=0$.

**Theorem 10.** *For every $c_1>0$, there exists $c_0>0$ such that the pair $(u,s):=(u^{c_0},s^{c_0})$, where $(u^{c_0},s^{c_0})$ come from Lemma 9 with $R=c_0$, satisfies the system (1). Furthermore,*

$$\forall x>0 \;\; u(x,\cdot),u_t(x,\cdot),u_x(x,\cdot)\in C([s^{-1}(x),\infty)), \tag{24}$$

$$\forall t>0 \;\; u(\cdot,t),u_t(\cdot,t)\in C([0,s(t)]), \;\; u_x(\cdot,t)\in C((0,s(t)]) \tag{25}$$

*and*

$$\forall t>0 \;\; \frac{\partial}{\partial x}D^\alpha u(\cdot,t)\in C([0,s(t)]). \tag{26}$$

*Finally, $u>0$, $u_t>0$, $u_x<0$ on $\{(x,t):\; 0<x<s(t),\; 0<t<\infty\}$.*

*Proof.* Let us show that there exists $c_0 > 0$ such that the pair $(u^R, s^R)$ given by Lemma 9 with $R = c_0$ satisfies $u^R(s^R(t),t) = 0$. For $\xi = xt^{-\frac{1}{1+\alpha}}$, the function $u^R$ defined in (22) is given by

$$u^R(x,t) = F(\xi) = c_1 + \tilde{c}_2 g(\xi),$$

where

$$g(\xi) = \left[\frac{\xi^{\alpha}}{\Gamma(\alpha+1)} + \xi^{\alpha}\sum_{k=1}^{\infty}\left(\frac{-\xi^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^k(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}\right].$$

We note that $g(0) = 0$, and since $\tilde{c}_2 < 0$, by Lemma 8 we infer that $g$ is increasing. Applying Lemma 6, we obtain that $\frac{\partial}{\partial x}D^{\alpha}g \leq 0$. Recalling that $\tilde{c}_2$ is given by (23), we arrive at

$$F(\xi) = c_1 - \frac{Rg(\xi)}{(\alpha+1)D^{\alpha}g(R)}.$$

We would like to find $R > 0$ such that $F(R) = 0$. We note that

$$F(R) = c_1 - \frac{Rg(R)}{(\alpha+1)D^{\alpha}g(R)}.$$

Since the denominator is positive, it is enough to show that there exists a positive zero of the function

$$h(R) := c_1(\alpha+1)D^{\alpha}g(R) - Rg(R).$$

We note that since $D^{\alpha}g(0) = 1$ we have $h(0) = c_1(\alpha+1) > 0$. On the other hand, since $g$ is absolutely continuous and $g(0) = 0$, we may write $g(R) = I^{\alpha}D^{\alpha}g(R)$. Applying $\frac{\partial}{\partial x}D^{\alpha}g \leq 0$, we may estimate as follows:

$$I^{\alpha}D^{\alpha}g(R) = \frac{1}{\Gamma(\alpha)}\int_0^R (R-p)^{\alpha-1}D^{\alpha}g(p)dp \geq \frac{D^{\alpha}g(R)}{\Gamma(\alpha)}\int_0^R (R-p)^{\alpha-1}dp = \frac{D^{\alpha}g(R)R^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence,

$$h(R) = c_1(1+\alpha)D^{\alpha}g(R) - RI^{\alpha}D^{\alpha}g(R) \leq c_1(1+\alpha)D^{\alpha}g(R) - RD^{\alpha}g(R)\frac{R^{\alpha}}{\Gamma(\alpha+1)}.$$

Recalling that $D^{\alpha}g > 0$, we arrive at $h(R) \to -\infty$ as $R \to \infty$. Hence, since $h$ is continuous, we may apply the Darboux property to deduce that there exists $c_0 > 0$ such that $h(c_0) = 0$, which implies $F(c_0) = 0$. Moreover, for $s(t) = c_0 t^{\frac{1}{1+\alpha}}$ there holds $u(s(t),t) = u(c_0 t^{\frac{1}{1+\alpha}}, t) = F(c_0) = 0$. The regularity results (24) and (25) immediately follow from (9) and regularity of $F$ established in Lemma 7. To show (26), we note that since $F$ satisfies (11), the continuity of $\xi F(\xi)$ implies $\frac{\partial}{\partial x}D^{\alpha}F \in C([0,R])$. This, together with identity (10), leads to (26). $\square$

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# STABILITY OF STATIONARY SOLUTIONS TO EQUATIONS DESCRIBING INCOMPRESSIBLE HEAT-CONDUCTING MOTIONS

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**Abstract:** We consider equations describing motion of an incompressible heat-conducting fluid in a bounded domain. We assume the slip boundary conditions for velocity and the Dirichlet condition for temperature. First, we prove the existence of a strong-weak stationary solution which is unique under some assumptions about the smallness of the data. Next, we show the existence of a global strong-weak solution to nonstationary problem which is close to the stationary solution. This way the stability of the strong-weak stationary solution in a set of strong-weak solutions to the nonstationary problem is proved.
**Keywords:** incompressible heat-conducting fluid, stability of stationary solution, global existence of strong-weak solution, slip boundary conditions
**Mathematics Subject Classification (2020):** 35B35, 35Q30, 76D03, 76D05

## 1. INTRODUCTION

The paper is concerned with equations describing incompressible viscous heat-conducting motions in a bounded domain $\Omega \subset \mathbb{R}^3$. We study the problem which consists of the Navier–Stokes equations coupled with the heat equation. The system is complemented with the slip boundary conditions for velocity and the Dirichlet condition for temperature. Thus, the problem under consideration is as follows

$$
\begin{aligned}
v_t - \nu\Delta v + v\cdot\nabla v + \nabla p = \alpha(\theta)f \quad & \text{in } \Omega\times\mathbb{R}_+,\\
\operatorname{div} v = 0 \quad & \text{in } \Omega\times\mathbb{R}_+,\\
\theta_t - \varkappa\Delta\theta + v\cdot\nabla\theta = \nu|\mathbb{D}(v)|^2 \quad & \text{in } \Omega\times\mathbb{R}_+,\\
\nu\bar{n}\mathbb{D}(v)\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, \quad & \text{on } S\times\mathbb{R}_+,\\
v\cdot\bar{n} = 0, \quad \theta = \bar{\theta} \quad & \text{on } S\times\mathbb{R}_+,\\
v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0 \quad & \text{in } \Omega,
\end{aligned}
\tag{1}
$$

where $S = \partial\Omega$, $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ is the velocity of the fluid, $x = (x_1,x_2,x_3)$ are the Cartesian coordinates, $p = p(x,t)$ is the pressure, $\theta = \theta(x,t)$ is the temperature of the fluid, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t))$ is the external force field, $\alpha\in C^1(\mathbb{R})$, $\bar{\theta} = \bar{\theta}(x)$ is a positive function defined on $\Omega$ (we assume that $\bar{\theta}\in H^1(\Omega)$), $\nu > 0$ is the viscosity coefficient, $\varkappa$ is the constant conductivity coefficient. Moreover, $\mathbb{D}(v) = \{v_{ix_j} + v_{jx_i}\}_{i,j=1,2,3} = \nabla v + \nabla v^T$ denotes the double velocity deformation tensor, $\bar{n}$ is the unit outward vector normal to $S$ and $\bar{\tau}_\alpha$, $\alpha = 1,2$ are tangent vectors to $S$ such that $\bar{n}$, $\bar{\tau}_\alpha$, $\alpha = 1,2$ form an orthonormal basis in $\mathbb{R}^3$.

Our aim is to prove the existence of a global strong-weak solution to problem (1) which is close to a strong-weak stationary solution. By a stationary solution to (1) we mean functions $w = w(x) = (w_1(x), w_2(x), w_3(x))$, $q = q(x)$ and $\vartheta = \vartheta(x)$ which satisfy the problem

$$
\begin{aligned}
-\nu\Delta w + w\cdot\nabla w + \nabla q = \alpha(\vartheta)g \quad & \text{in } \Omega,\\
\operatorname{div} w = 0 \quad & \text{in } \Omega,\\
-\varkappa\Delta\vartheta + w\cdot\nabla\vartheta = \nu|\mathbb{D}(w)|^2 \quad & \text{in } \Omega,\\
\nu\bar{n}\mathbb{D}(w)\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, \quad & \text{on } S,\\
w\cdot\bar{n} = 0, \quad \vartheta = \bar{\theta} \quad & \text{on } S.
\end{aligned}
\tag{2}
$$

In order to obtain the main result of the paper we introduce the functions $u = v - w$, $\eta = p - q$, $\chi = \theta - \vartheta$, $h = f - g$ which are solutions to the problem

$$
\begin{aligned}
& u_t - \nu\Delta u + u\cdot\nabla u + \nabla\eta \\
& \quad = -w\cdot\nabla u - u\cdot\nabla w + [\alpha(\chi+\vartheta) - \alpha(\vartheta)]f + \alpha(\vartheta)h \quad && \text{in } \Omega\times\mathbb{R}_+,\\
& \operatorname{div} u = 0 \quad && \text{in } \Omega\times\mathbb{R}_+,\\
& \chi_t - \varkappa\Delta + u\cdot\nabla\chi \\
& \quad = -w\cdot\nabla\chi - u\cdot\nabla\vartheta + \nu|\mathbb{D}(u)|^2 + 2\nu\mathbb{D}(u):\mathbb{D}(w) \quad && \text{in } \Omega\times\mathbb{R}_+,\\
& \nu\bar{n}\mathbb{D}(u)\bar{\tau}_\alpha = 0, \quad \alpha = 1,2, \quad && \text{on } S\times\mathbb{R}_+,\\
& u\cdot\bar{n} = 0, \quad \chi = 0 \quad && \text{on } S\times\mathbb{R}_+,\\
& u|_{t=0} = v_0 - w, \quad \chi|_{t=0} = \theta_0 - \vartheta.
\end{aligned}
\tag{3}
$$

In Section 2 we present notation and the main results of the paper which are formulated as Theorems 3, 8. Theorems 3–4 are concerned with the unique solvability of problem (2), and Theorems 7–8 contain results concerning stability of a stationary solution and the existence of a strong-weak solution to problem (1). In Section 3 we formulate auxiliary results used in the proofs of the main theorems. In Section 4 we present the proofs of Theorems 3 and 4 concerning the uniqueness of a stationary solution. Finally, in Section 5 we derive a differential inequality for a solution to problem (3). Next, using this inequality together with the Galerkin approximations we prove Theorems 7 and 8.

To prove Theorems 3, 8 we use methods similar to those from paper [15] in which equations $(1)_{1,2,3}$ complemented with the Dirichlet boundary condition both for velocity and temperature were considered. The temperature on the boundary was assumed constant. The applied methods are adapted to the case of the slip boundary conditions for velocity and temperature which is not constant on the boundary.

In [13] and [14] system $(1)_{1,2,3}$ is also studied. Both papers are concerned with the initial-boundary value problem in a cylinder complemented with the slip boundary or Navier's condition for velocity. The stability of a two-dimensional solution to the problem in a set of three-dimensional solutions is studied. Moreover, the existence of a global strong-weak solution to problem (1) close to the two-dimensional solution is proved.

I. Kagei examined in [6] and [7] the existence, uniqueness and large time behaviour for the two-dimensional system $(1)_{1,2,3}$, where the left-hand side of equation $(1)_3$ additionally contains the term $-e_2 \cdot v$, $e_2 = (0,1)$.

The stability of a stationary solution to the Navier–Stokes system with the nonhomogeneous Dirichlet boundary condition is studied in [8]. The existence of a weak solution to the nonstationary problem which tends to a solution of the stationary problem as $t \to \infty$, is proved.

Moreover, [2]–[5] and [9]–[11] are devoted to various solvability and large time behaviour questions for the Boussinesq system, that is, the system of equations $(1)_{1,2,3}$, where the term $\nu|\mathbb{D}(v)|^2$ disappears.

## 2. RESULTS

Before stating the results we introduce the following notation. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\Omega \subset \mathbb{R}^3$ be an open set. Norms in the Lebesgue spaces $L_p(\Omega)$, $p \in [1,\infty]$ and in the Sobolev spaces $W_p^m(\Omega)$, $p \in [1,\infty]$ are denoted by $\|\cdot\|_{L_p}$ and $\|\cdot\|_{W_p^m}$, respectively. In the special case of the space $H^m(\Omega) = W_2^m(\Omega)$ the norm is denoted by $\|\cdot\|_{H^m}$. We also use the notation:

$$V = \{u \in H^1(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \bar{n} = 0 \text{ on } S\}.$$

Let $I \subset \mathbb{R}$ be an open interval. Then $H^{2,1}(\Omega \times I)$ denotes the space of functions $u$ with the norm

$$\|u\|_{H^{2,1}(\Omega\times I)} = \Big(\|u_t\|^2_{L_2(\Omega\times I)} + \sum_{0\le|\alpha|\le 2} \|D_x^\alpha u\|^2_{L_2(\Omega\times I)}\Big)^2,$$

where $D_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha_i \in \mathbb{N}_0$, $i = 1, \ldots, n$.

Let $X$ be a Banach space. By $L_p(I;X)$ we denote the space of all measurable functions $u: I \to X$ with the norm

$$\|u\|_{L_p(I;X)} = \left( \int_I \|u(t)\|_X^p \, dt \right)^{1/p} \quad \text{if } 1 < p < \infty$$

and

$$\|u\|_{L_\infty(I;X)} = \operatorname{ess\,sup}_{t \in I} \|u(t)\|_x.$$

Moreover, $C(\tilde{I};X)$ denotes the space of all continuous functions $u: \tilde{I} \to X$ with the norm $\|u\|_{C(\tilde{I};X)} = \sup_{t \in \tilde{I}} \|u(t)\|_x$.

First, we will formulate the main results concerning the stationary problem (2). We assume that $\bar{\theta} \in H^1(\Omega)$ and introduce the function $\hat{\vartheta} = \vartheta - \bar{\theta}$. Then problem (2) takes the form

$$\begin{aligned}
-\nu\Delta w + w \cdot \nabla w + \nabla q &= \alpha(\hat{\vartheta} + \bar{\theta})g && \text{in } \mathbb{R}, \\
\operatorname{div} u &= 0 && \text{in } \Omega, \\
-\varkappa\Delta\hat{\vartheta} + w \cdot \nabla\hat{\vartheta} &= \nu|\mathbb{D}(w)|^2 + \varkappa\Delta\bar{\theta} - w \cdot \nabla\bar{\theta} && \text{in } \Omega, \\
\nu\bar{n}\mathbb{D}(w)\bar{\tau}_\alpha = 0, \quad \alpha &= 1,2, && \text{on } S, \\
w \cdot \bar{n} &= 0 && \text{on } S, \\
\hat{\vartheta} &= 0 && \text{on } S.
\end{aligned} \tag{4}$$

**Definition 1.** *We call a function* $(w, \hat{\vartheta}) \in V \times H_0^1(\Omega)$ *a weak solution to problem* (4) *if the following integral identities hold*

$$\frac{\nu}{2} \int_\Omega \mathbb{D}(w) : \mathbb{D}(\psi)\, dx + \int_\Omega w \cdot \nabla w \psi \, dx = \int_\Omega \alpha(\hat{\vartheta} + \bar{\theta}) g \psi \, dx \quad \forall \psi \in V \tag{5}$$

*and*

$$\begin{aligned}
&\varkappa \int_\Omega \nabla\hat{\vartheta} \cdot \nabla\varphi \, dx + \int_\Omega w \cdot \nabla\hat{\vartheta}\varphi \, dx \\
&\qquad = \nu \int_\Omega |\mathbb{D}(w)|^2 \varphi \, dx + \varkappa \int_\Omega \nabla\bar{\theta} \cdot \nabla\varphi \, dx - \int_\Omega w \cdot \nabla\bar{\theta}\varphi \, dx \quad \forall \varphi \in H_0^1(\Omega),
\end{aligned} \tag{6}$$

*where* $\mathbb{D}(w) : \mathbb{D}(\psi) = \sum_{i,j=1}^3 D_{ij}(w) D_{ij}(\psi)$, $w \cdot \nabla w \psi = \sum_{i,j=1}^3 w_i w_{jx_i} \psi_j$, $g\psi = \sum_{i=1}^3 g_i \psi_i$.

**Definition 2**

1° *We say that a function* $(w, q, \hat{\vartheta})$ *is a strong-weak solution to* (4) *if* $(w, \hat{\vartheta})$ *is a weak solution to* (4), $(w, q) \in H^2(\Omega) \times H^1(\Omega)$ *with* $\int_\Omega q \, dx = 0$ *and* $(w, q, \hat{\vartheta})$ *satisfies equation* $(4)_1$ *almost everywhere in* $\Omega$.

2° *A function* $(w, q, \vartheta)$ *is called a strong-weak solution to* (2) *if* $(w, q, \hat{\vartheta})$ *is a strong-weak solution to* (4)*, where* $\hat{\vartheta} = \vartheta - \bar{\theta}$.

Our results referred to the stationary problem read as follows.

**Theorem 3.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $S \in C^3$. Let $g \in L_\infty(\Omega)$, $\bar{\theta} \in H^1(\Omega)$, $0 < \sigma < 1/8$, $\alpha \in C^1(\mathbb{R})$, $|\alpha(\vartheta)| \le a_1 + a_2|\vartheta|^\sigma$, $|\alpha'(\vartheta)| \le a_3$ for $\vartheta \in \mathbb{R}$, where $a_i$, $i = 1,2,3$, are constants such that $a_1 \ge 0$, $a_2, a_3 > 0$, Assume that $\bar{\theta} \ge \theta_*$ almost everywhere in $\Omega$, where $\theta_*$ is a positive constant. Then there exists a strong-weak solution $(w,q,\vartheta) \in H^2(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ with $\int_\Omega q(x)\,dx = 0$ to problem* (2). *Moreover,*

$$\|w\|_{H^2} + \|q\|_{H^1} + \|\vartheta - \bar{\theta}\|_{H^1} \le \varphi(\|g\|_{L_\infty} + \|\bar{\theta}\|_{H^1})(\|g\|_{L_\infty} + \|\nabla\bar{\theta}\|_{L_2}),$$

*where $\varphi = \varphi(\|g\|_{L_\infty} + \|\bar{\theta}\|_{H^1})$ is a continuous increasing function.*

**Theorem 4.** *Let the assumptions of Theorem* 3 *hold. Assume that*

$$\|g\|_{L_\infty} + \|\nabla\bar{\theta}\|_{L_2} \le \delta_1,$$

*where $\delta_1 > 0$ is a sufficiently small constant. Then there exists a unique strong-weak solution to problem* (2).

Now, we formulate results concerning the nonstationary problem. First, we define a weak and strong-weak solution to problem (1).

**Definition 5.** *Let $T > 0$ be given and let $(w, \vartheta, q)$ be a strong-weak solution of problem* (2). *We call a function $(u, \chi)$ a weak solution to problem* (3) *if*

$$\begin{aligned}
&u \in L_\infty(kT,(k+1)T;L_2(\Omega)) \cap L_2(kT,(k+1)T;V), && u_t \in L_2(kT,(k+1)T;V^*),\\
&\chi \in L_\infty(kT,(k+1)T;L_2(\Omega)) \cap L_2(kT,(k+1)T;H_0^1(\Omega)), && \chi_t \in L_2(kT,(k+1)T;H^{-1}(\Omega))
\end{aligned}$$

*for all $k \in \mathbb{N}_0$ and*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u\psi\,dx &+ \frac{\nu}{2}\int_\Omega \mathbb{D}(u) : \mathbb{D}(\psi)\,dx + \int_\Omega u\cdot\nabla u\psi\,dx\\
&= -\int_\Omega w\cdot\nabla u\psi\,dx - \int_\Omega u\cdot\nabla w\psi\,dx + \int_\Omega (\alpha(\vartheta+\chi) - \alpha(\vartheta)) f\psi\,dx\\
&\quad + \int_\Omega \alpha(\vartheta) h\psi\,dx \qquad \forall \psi \in V
\end{aligned}$$

*in the sense of distributions on $(kT,(k+1)T)$,*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \chi\varphi\,dx &+ \varkappa\int_\Omega \nabla\chi\cdot\nabla\varphi\,dx + \int_\Omega u\cdot\nabla\chi\varphi\,dx\\
&= -\int_\Omega w\cdot\nabla\chi\varphi\,dx - \int_\Omega u\cdot\nabla\vartheta\varphi\,dx\\
&\quad + \nu\int_\Omega |\mathbb{D}(u)|^2\varphi\,dx + 2\nu\int_\Omega \mathbb{D}(u) : \mathbb{D}(w)\varphi\,dx \qquad \forall \varphi \in H_0^1(\Omega)
\end{aligned}$$

*in the sense of distributions on $(kT,(k+1)T)$,*

$$\begin{aligned}
u|_{t=kT} &= u(kT),\\
\chi|_{t=kT} &= \chi(kT)
\end{aligned}$$

*for all $k \in \mathbb{N}_0$.*

**Remark.** *In the above definition* $u|_{t=0} = u(0) \equiv v_0 - w$, $\chi|_{t=0} = \chi(0) \equiv \theta_0 - \vartheta$ *and the initial conditions* $u|_{t=kT} = u(kT)$, $\chi|_{t=kT} = \chi(kT)$ *for* $k \in \mathbb{N}$ *mean that the initial data at the point* $t = kT$ *are the terminal values* $u(kT)$ *and* $\chi(kT)$ *of the functions* $u$ *and* $\chi$ *in the interval* $[(k-1)T, kT]$, $k \in \mathbb{N}$.

**Definition 6**

1° *The triple* $(u, \chi, \eta)$ *is called a strong-weak solution to problem* (3) *if* $(u, \chi)$ *is a weak solution to* (3) *with initial conditions* $u|_{t=kT} = u(kT), \chi|_{t=kT} = \chi(kT)$ *for all* $k \in \mathbb{N}_0$, $(u, \eta) \in (L_\infty(kT, (k+1)T; H^1(\Omega)) \cap L_2(kT, (k+1)T; H^2(\Omega))) \times L_2(kT, (k+1)T; H^1(\Omega))$ *with* $\int_\Omega \eta\, dx = 0$ *and if* $(u, \alpha, \eta)$ *satisfies* $(3)_1$ *almost everywhere in* $\Omega \times (kT, (k+1)T)$ *for all* $k \in \mathbb{N}_0$.

2° *A function* $(v, \theta, p)$ *is called a strong-weak solution to* (1) *if* $(u, \chi, \eta)$ *is a strong-weak solution to* (3).

Our next result is concerned with the stability of a strong-weak stationary solution under small nonstationary perturbations.

**Theorem 7.** *Let the assumption of Theorem* 4 *hold. Let* $v_0 \in V$, $\theta_0 \in L_2(\Omega)$, $\theta_0 \geq \theta_*$ *almost everywhere in* $\Omega$ *and let* $T > 0$ *be given. Assume that* $f \in C(\mathbb{R}_+; L_\infty(\Omega))$ *and*

$$\sup_{k \in \mathbb{N}_0} \|f\|_{C([kT,(k+1)T]; L_\infty)} \leq \delta_1. \tag{7}$$

*Moreover, let*

$$\|\mathbb{D}(u(0))\|_{L_2}^2 + \|u(0)\|_{L_2}^2 + \|\chi(0)\|_{L_2}^2 \leq \gamma \tag{8}$$

*and*

$$\|h(t)\|_{L_4}^2 \leq \delta_2 \gamma \quad \text{for all } t \in \mathbb{R}_+ \tag{9}$$

*where* $\delta_2, \gamma > 0$ *are some constants. Let* $(w, \vartheta, q)$ *be the strong-weak solution to problem* (2) *which exists in virtue of Theorems* 3 *and* 4. *Assume that* $(u, \chi, \eta)$ *is a strong-weak solution to problem* (3). *If* $\gamma$ *and* $\delta_i$ $(i = 1, 2)$ *are sufficiently small then*

$$\|\mathbb{D}(u(t))\|_{L_2}^2 + \|u(t)\|_{L_2}^2 + \|\chi(t)\|_{L_2}^2 \leq \gamma \quad \text{for all } t \in \mathbb{R}_+. \tag{10}$$

*Moreover,*

$$\begin{aligned} \|u\|^2_{H^{2,1}(\Omega \times (kT,(k+1)T))} &+ \|\chi\|^2_{L_2(kT,(k+1)T;H^1)} \\ &+ \|\chi_t\|^2_{L_2(kT,(k+1)T;H^{-1})} + \|\nabla \eta\|^2_{L_2(kT,(k+1)T;L_2)} \leq c(T)\gamma, \end{aligned} \tag{11}$$

*where* $c = c(T)$ *does not depend on* $k$.

To prove the global existence theorem for problem (1) we apply the Galerkin approximations. We get

**Theorem 8.** *Let the assumptions of Theorem* 4 *hold. Let* $v_0 \in V$, $\theta_0 \in L_2(\Omega)$, $\theta_0 \geq \theta_*$ *almost everywhere in* $\Omega$ *and let* $T > 0$ *be given. Assume that* $f \in C(\mathbb{R}_+; L_\infty(\Omega))$ *and conditions* (7)–(9) *are satisfied. Let* $(w, \vartheta, q)$ *be the strong-weak solution to problem* (2) *which exists in virtue of Theorems* 4 *and* 5. *If* $\gamma$ *and* $\delta_i$ $(i = 1,2)$ *are sufficiently small then there exists a unique strong-weak global solution to problem* (1)

$$\begin{aligned}(v, \theta, p) \in H^{2,1}&(\Omega \times (kT, (k+1)T)) \\ &\times (C([kT, (k+1)T]; L_2(\Omega)) \cap L_2(kT, (k+1)T; H^1(\Omega))) \\ &\qquad \times L_2(kT, (k+1)T; H^1(\Omega)), \qquad (k \in \mathbb{N}_0),\end{aligned}$$

*with* $\int_\Omega p\, dx = 0$. *Moreover,* $\theta(kT) \geq \theta_*$ *almost everywhere in* $\Omega$ *for* $k \in \mathbb{N}_0$.

# 3. AUXILARY RESULTS

In what follows we use the following lemmas.

**Lemma 9.** *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain and* $T > 0$. *Let* $\theta_0 \in L_2(\Omega)$ *and assume that* $\theta_0 \geq \theta_*$ *almost everywhere in* $\Omega$ *and* $\bar{\theta} \geq \theta_*$ *almost everywhere in* $\Omega$, *where* $\theta_*$ *is a positive constant. Let* $(v, \theta, p)$ *be a strong-weak solution to problem* (1). *Then*

$$\theta \geq \theta_* \quad \textit{almost everywhere in } \Omega \times \mathbb{R}_+.$$

*Proof.* Notice that the following identity holds:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\theta - \theta_*)\varphi\, dx + \varkappa \int_\Omega \nabla(\theta - \theta_*) \cdot \nabla \varphi\, dx \\ + \int_\Omega v \cdot \nabla(\theta - \theta_*)\varphi\, dx = \nu \int_\Omega |\mathbb{D}(v)|^2 \varphi\, dx \quad \forall \varphi \in H_0^1(\Omega).\end{aligned}$$

Since $(\theta - \theta_*)_- = (\bar{\theta} - \theta_*)_- = 0$ on $S$, inserting $\varphi = (\theta - \theta_*)_- = \min(\theta - \theta_*, 0)$ yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\theta - \theta_*)_-^2\, dx + \varkappa \int_\Omega |\nabla(\theta - \theta_*)_-|^2\, dx = \nu \int_\Omega |\mathbb{D}(v)|^2 (\theta - \theta_*)_-\, dx \leq 0.$$

Hence,

$$\int_\Omega (\theta(t) - \theta_*)_-^2\, dx \leq \int_\Omega (\theta_0 - \theta_*)_-^2\, dx = 0,$$

so

$$\theta \geq \theta_* \quad \text{almost everywhere in } \Omega \times \mathbb{R}_+. \qquad \square$$

Consider the following stationary Stokes system with the slip boundary conditions

$$\begin{aligned} \operatorname{div} \mathbb{T}(v,p) &= h && \text{in } \Omega, \\ \operatorname{div} v &= 0 && \text{in } \Omega, \\ v \cdot \bar{n} &= 0 && \text{on } S, \\ \nu \bar{n} \mathbb{D}(v) \bar{\tau}_\alpha &= 0,\ \alpha = 1,2, && \text{on } S. \end{aligned} \tag{12}$$

**Lemma 10.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $S \in C^2$ and let $(v,p)$ be a weak solution to problem* (12). *Then $v \in H^2(\Omega)$, $\nabla p \in L_2(\Omega)$ and*

$$\|v\|_{H^2} + \|\nabla p\|_{L_2} \le c\|h\|_{L_2}.$$

The assertion of Lemma 10 follows from the regularity theory for general elliptic equations and systems (see [1], [12]).

# 4. EXISTENCE OF A STATIONARY SOLUTION

The purpose of this section is to present proofs of Theorems 3 and 4. To prove Theorem 3 we need the following lemma:

**Lemma 11.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $S \in C^2$. Let $0 < \sigma < 1/8$, $\alpha \in C(\mathbb{R})$ and*

$$|\alpha(\vartheta)| \le a_1 + a_2|\vartheta|^\sigma \quad \textit{for } \vartheta \in \mathbb{R}, \tag{13}$$

*where $a_i$, $i = 1,2$, are constants such that $a_1 \ge 0$, $a_2 > 0$. Let $g \in L_\infty(\Omega)$, $p_1 = \frac{4}{3\sigma}$, $p_2 = \frac{1}{\sigma}$, $r = \frac{1}{8\sigma}$, $s_1 = \frac{1}{4\sigma}$, $s_2 = \frac{1}{2\sigma}$, $p_i' = \frac{p_i}{p_i - 1}$, $s_i' = \frac{s_i}{s_i - 1}$, $i = 1,2$, $r' = \frac{r}{r-1}$. Assume that $(w,q,\hat{\vartheta})$ is a strong-weak solution to problem* (4). *Then*

$$\|w\|_{H^2} + \|q\|_{H^1} + \|\hat{\vartheta}\|_{H^1} \le cF, \tag{14}$$

*where*

$$\begin{aligned} F = {} & (\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2 + \|g\|_{L_\infty}^2 \|\bar{\theta}\|_{L_2}^{2\sigma} + \|g\|_{L_{3/2}} + \|\bar{\theta}\|_{L_{\frac{3}{2}\sigma p_1}}^\sigma \|g\|_{L_{\frac{3}{2}p_1'}} + \|g\|_{L_{\frac{3}{2}p_1'}}^{s_1'})^2 \\ & + \|g\|_{L_2} + \|g\|_{L_{2p_2'}}^{s_2'} + \|\nabla\bar{\theta}\|_{L_2}^2 + \|\nabla\bar{\theta}\|_{L_2}. \end{aligned}$$

*Proof.* The proof is similar to the proof of [15, Lemma 3.2]. Inserting $\psi = w \in V \cap H^2(\Omega)$ into (5) we obtain

$$\frac{\nu}{2}\|\mathbb{D}(w)\|_{L_2}^2 = \int_\Omega \alpha(\hat{\vartheta} + \bar{\theta}) g w \, dx.$$

By the Korn inequality we obtain

$$\|w\|_{H^1}^2 \le c\big[\|g\|_{L_\infty}\|w\|_{L_q}(\|\hat{\vartheta}\|_{L_{\sigma q'}}^\sigma + \|\bar{\theta}\|_{L_{\sigma q'}}^\sigma) + \|g\|_{L_2}\|w\|_{L_2}\big],$$

where $q' = \frac{2}{\sigma} > 16$, $q = \frac{q'}{q'-1}$. Hence by the Poincaré inequality

$$\|w\|_{H^1} \le c\big[\|g\|_{L_\infty}(\|\hat{\vartheta}\|_{L_2}^{\sigma} + \|\bar{\theta}\|_{L_2}^{\sigma}) + \|g\|_{L_2}\big]$$

and therefore

$$\|w\|_{H^1}^2 \le \varepsilon\|\hat{\vartheta}\|_{L_2}^{1/4} + c(\varepsilon)(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2) + c\|g\|_{L_\infty}^2\|\bar{\theta}\|_{L_2}^{2\sigma}, \tag{15}$$

where $\varepsilon > 0$.

Next, inserting $\varphi = \hat{\vartheta}$ into (6) we get

$$\begin{aligned}\varkappa\|\nabla\hat{\vartheta}\|_{L_2}^2 &= \nu\int_\Omega |\mathbb{D}(w)|^2\hat{\vartheta}\,dx + \varkappa\int_\Omega \nabla\bar{\theta}\cdot\nabla\hat{\vartheta}\,dx - \int_\Omega w\cdot\nabla\bar{\theta}\hat{\vartheta}\,dx\\ &\le c(\|w\|_{H^2}^2 + \|\nabla\bar{\theta}\|_{L_2} + \|\nabla\bar{\theta}\|_{L_2}^2)\|\nabla\hat{\vartheta}\|_{L_2},\end{aligned}$$

which yields

$$\|\hat{\vartheta}\|_{H^1} \le c(\|w\|_{H^2}^2 + \|\nabla\bar{\theta}\|_{L_2}^2 + \|\nabla\bar{\theta}\|_{L_2}). \tag{16}$$

We need an estimate of $\|w\|_{L_2}$. Therefore we rewrite $(4)_{1,2,4,5}$ in the form

$$\begin{aligned} -\nu\Delta w + \nabla q &= \alpha(\hat{\vartheta} + \bar{\theta})g - w\cdot\nabla w &&\text{in } \Omega,\\ \operatorname{div} w &= 0 &&\text{in } \Omega,\\ \nu\bar{n}\mathbb{D}(w)\bar{\tau}_\alpha &= 0, \quad \alpha = 1,2, &&\text{on } S,\\ w\cdot\bar{n} &= 0 &&\text{on } S.\end{aligned}$$

By [1], [12] it follows the inequality

$$\|w\|_{W_\delta^2} + \|\nabla q\|_{L_\delta} \le c(\|\alpha(\hat{\vartheta} + \bar{\theta})g\|_{L_\delta} + \|w\cdot\nabla w\|_{L_\delta}). \tag{17}$$

Let $\delta = 3/2$. Then (15) gives

$$\|w\cdot\nabla w\|_{L_{3/2}} \le c\|w\|_{H^1}^2 \le c(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2) + \varepsilon\|\hat{\vartheta}\|_{L_2}^{1/4} + c\|g\|_{L_\infty}^2\|\bar{\theta}\|_{L_2}^{2\sigma}.$$

Moreover,

$$\|\alpha(\hat{\vartheta} + \bar{\theta})g\|_{L_{3/2}} \le c(\|g\|_{L_{3/2}} + \|\hat{\vartheta}\|_{L_{\frac{3}{2}\sigma p_1}}^{\sigma}\|g\|_{L_{\frac{3}{2}p_1'}} + \|\bar{\theta}\|_{L_{\frac{3}{2}\sigma p_1}}^{\sigma}\|g\|_{L_{\frac{3}{2}p_1'}}).$$

Hence,

$$\begin{aligned}&\|w\|_{W_{3/2}^2} + \|\nabla q\|_{L_{3/2}}\\ &\quad\le \varepsilon\|\hat{\vartheta}\|_2^{1/4} + c\big(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2 + \|g\|_{L_\infty}^2\|\bar{\theta}\|_{L_2}^{2\sigma} + \|g\|_{L_{3/2}} + \|\bar{\theta}\|_{L_{\frac{3}{2}\sigma p_1}}^{\sigma}\|g\|_{L_{\frac{3}{2}p_1'}} + \|g\|_{L_{\frac{3}{2}p_1'}}^{s_1'}\big).\end{aligned}$$

Next, we use (17) with $\delta = 2$. Proceding in the same way as in [15] we get

$$\begin{aligned}\|w\|_{H^2} + \|\nabla q\|_{L_2} \le \varepsilon\|\hat{\vartheta}\|_{L_2}^{1/2} + c\big[(\|g\|_{L_\infty}^{2r'} + \|g\|_{L_2}^2 + \|g\|_{L_\infty}^2\|\bar{\theta}\|_{L_2}^{2\sigma} + \|g\|_{L_{3/2}}\\ + \|\bar{\theta}\|_{L_{\frac{3}{2}\sigma p_1}}^{\sigma}\|g\|_{L_{\frac{3}{2}p_1'}} + \|g\|_{L_{\frac{3}{2}p_1'}}^{s_1'})^2 + \|g\|_{L_2} + \|g\|_{L_{2p_2'}}^{s_2'}\big].\end{aligned} \tag{18}$$

Using (13) in (18) and assuming that $\varepsilon$ is sufficiently small we get (14). □

Now, we consider problem $(4)_{3,6}$ for given $w$.

**Lemma 12.** *For given $w \in H^2(\Omega) \cap V$ and $\bar\theta \in H^1(\Omega)$ there exists a unique weak solution $\hat\vartheta \in H^1_0(\Omega)$ of the problem*

$$\begin{aligned} -\varkappa\Delta\hat\vartheta + w\cdot\nabla\hat\vartheta &= \nu|\mathbb{D}(w)|^2 + \varkappa\Delta\bar\theta - w\cdot\nabla\bar\theta \quad && \text{in } \Omega, \\ \hat\vartheta &= 0 && \text{on } S. \end{aligned} \tag{19}$$

*Moreover, the solution satisfies inequality* (16).

Lemma 12 follows by applying the Lax–Milgram theorem.

Now, let us fix $w \in H^2(\Omega)\cap V$ and introduce an operator $T: H^2(\Omega)\cap V \to H^2(\Omega)\cap V$ such that $Tw = w^*$ given by the formula

$$\frac{\nu}{2}\int_\Omega \mathbb{D}(w^*) : \mathbb{D}(\psi)\,dx + \int_\Omega w\cdot\nabla w\psi\,dx = \int_\Omega \alpha(\hat\vartheta + \bar\theta)g\psi\,dx \quad \forall\psi\in V, \tag{20}$$

where $\hat\vartheta \in H^1_0(\Omega)$ is a weak solution to problem (19).

**Lemma 13.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary $S \in C^3$. Let $g \in L_4(\Omega)$, $\bar\theta \in H^1(\Omega)$, $0 < \sigma < 1/8$ and condition* (13) *hold. Moreover, let $|\alpha'(\vartheta)| \le a_3$ for $\vartheta \in \mathbb{R}$, where $a_3 > 0$ is a constant. Assume that for $i = 1,2$ we have $w_i \in H^2(\Omega)\cap V$, $w_i^* = Tw_i$, where $\hat\vartheta_i$ is the solution of* (19) *corresponding to $w_i$. Then*

$$\begin{aligned} \|w_1^* - w_2^*\|_{H^2} \le c\big[(\|\hat\vartheta_1\|_{H^1} + \|\nabla\bar\theta\|_{L_2})\|g\|_{L_4} \\ + (\|w_1\|_{W^1_4} + \|w_2\|_{W^1_4})(\|g\|_{L_4}+1)\big]\,\|w_1 - w_2\|_{W^1_4}. \end{aligned} \tag{21}$$

*Proof.* We substract (6) for $i=2$ from (6) for $i=1$ and then insert $\varphi = \hat\vartheta_1 - \hat\vartheta_2$. Using the Poincaré inequality we obtain

$$\|\hat\vartheta_1 - \hat\vartheta_2\|_{H^1} \le c(\|\hat\vartheta_1\|_{H^1} + \|w_1\|_{W^1_4} + \|w_2\|_{W^1_4} + \|\nabla\bar\theta\|_{L_2})\|w_1 - w_2\|_{W^1_4}. \tag{22}$$

Since $w_i^* \in H^2(\Omega)\cap V$, $i = 1,2$, satisfies (20) there exist $q_i \in H^1(\Omega)$, $i=1,2$, such that

$$\begin{aligned} &-\nu\Delta(w_1^* - w_2^*) + \nabla(q_1 - q_2) \\ &\qquad = (\alpha(\vartheta_1+\bar\theta) - \alpha(\vartheta_2+\bar\theta))g + w_2\cdot\nabla w_2 - w_1\cdot\nabla w_1 \quad && \text{in } \Omega, \\ &\qquad \operatorname{div}(w_1^* - w_2^*) = 0 && \text{in } \Omega, \\ &\qquad (w_1^* - w_2^*)\bar n = 0 && \text{on } S, \\ &\qquad \bar n\mathbb{T}(w_1^* - w_2^*, q_1 - q_2)\bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S, \end{aligned}$$

where by the mean value theorem

$$\begin{aligned} &|\alpha(\vartheta_1+\bar\theta) - \alpha(\vartheta_2+\bar\theta)| \\ &\qquad = |\alpha'(\beta(\hat\vartheta_1+\bar\theta) + (1-\beta)(\hat\vartheta_2+\bar\theta))|\,|\vartheta_1 - \vartheta_2| \le a_3|\vartheta_1 - \vartheta_2|, \quad \beta\in(0,1). \end{aligned}$$

Therefore, $w_1^* - w_2^*$ satisfies the inequality

$$\begin{aligned}\|w_1^* - w_2^*\|_{H^2} &+ \|\nabla(q_1 - q_2)\|_{L_2}\\ &\le c\big[\|(\hat{\vartheta}_1 - \hat{\vartheta}_2)g\|_{L_2} + \|(w_1 - w_2)\cdot\nabla w_1\|_{L_2} + \|w_2\cdot\nabla(w_1 - w_2)\|_{L_2}\big]\\ &\le c\big[\|\|\hat{\vartheta}_1 - \hat{\vartheta}_2\|_{H^1}\|g\|_{L_4} + \|w_1 - w_2\|_{W_4^1}(\|w_1\|_{W_4^1} + \|w_2\|_{W_4^1})\big].\end{aligned}$$

Hence in view of (22) inequality (21) follows. □

Now we can prove Theorems 3 and 4.

*Proof of Theorem* 3. By Lemma 13 the operator $T : H^2(\Omega)\cap V \to H^2(\Omega)\cap V$ is continuous and compact since the imbedding $H^2(\Omega)\cap V \to W_4^1(\Omega)$ is compact. Assume that $w \in H^2(\Omega)\cap V$ is a solution to the equation

$$w = \lambda Tw,$$

where $\lambda \in [0,1]$. Then $w$ satisfies (14). Therefore, by the Leray–Schauder fixed point theorem there exists a strong-weak solution to problem (4) which implies the existence of strong-weak solution of problem (2). □

*Proof of Theorem 4.* Let $(w_i, q_i, \hat{\vartheta}_i)$, $i = 1,2$, be two different solutions to problem (4). Repeating the proof of Lemma 13 we get the inequality

$$\begin{aligned}\|w_1 - w_2\|_{H^2} &+ \|\nabla(q_1 - q_2)\|_{L_2} + \|\hat{\vartheta}_1 - \hat{\vartheta}_2\|_{H^1}\\ &\le c\big[(\|\hat{\vartheta}_1\|_{H^1} + \|\nabla\bar{\theta}\|_{L_2} + \|w_1\|_{H^2} + \|w_2\|_{H^2})(1 + \|g\|_{L_4})\big]\|w_1 - w_2\|_{H^2}.\end{aligned}$$

By Lemma 11 it follows that

$$\|w_1\|_{H^2} + \|w_2\|_{H^2} + \|\hat{\vartheta}_1\|_{H^1} + \|\nabla\bar{\theta}\|_{L_2} \le cF.$$

Therefore if $\delta_1$ is sufficiently small then

$$cF(1 + \|g\|_{L_4}) < 1.$$

Thus, we get the uniqueness of problem (4). Therefore, the uniqueness of a solution $(w, q, \vartheta)$ to problem (2) also follows. □

## 5. STABILITY OF THE STATIONARY SOLUTION AND EXISTENCE OF A SOLUTION TO PROBLEM (1)

The aim of this section is to prove Theorems 7 and 8. First, we will derive differential inequality which is essential for the proof of stability of the stationary solution. We start with some lemmas.

**Lemma 14.** *Let the assumptions of Theorem* 3 *hold and let $T > 0$ be given. Let $f \in C([kT,(k+1)T];L_\infty(\Omega))$ for all $k \in \mathbb{N}_0$. Assume that $(w,q,\vartheta)$ is the stationary solution which exists in virtue of Theorem* 3. *Let $(u,\chi,\eta)$ be a sufficiently regular solution to problem* (3). *Then*

$$\begin{aligned}\frac{d}{dt}(\|u\|_{L_2}^2+\|\chi\|_{L_2}^2)+c\left(\frac{\nu}{2}\|\mathbb{D}(u)\|_{L_2}^2+\varkappa\|\chi\|_{H^1}^2\right)\\ \le c(\varepsilon)\big(\|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2+\|\nabla\vartheta\|_{L_2}^2\|u\|_{L_3}^2+\|u\|_{H^1}^2\|\nabla w\|_{L_3}^2\\ +\|h\|_{L_4}+\|h\|_{L_4}^2\|\vartheta\|_{L_2}+\|\chi\|_{L_2}^2\|f\|_{L_3}^2\big)+\varepsilon\|u\|_{H^2}^2 \quad \text{in } (kT,(k+1)T),\, k\in\mathbb{N}_0,\end{aligned} \tag{23}$$

*where $\varepsilon > 0$ is a constant and the constant $c = c(\varepsilon)$ does not depend on $k$.*

*Proof.* Let us rewrite equation $(3)_1$ in the form

$$\begin{aligned}u_t-\operatorname{div}\mathbb{T}(u,\eta)\\ =-(u\cdot\nabla)u-(u\cdot\nabla)w-w\cdot\nabla u+[\alpha(\chi+\vartheta)-\alpha(\vartheta)]f+\alpha(\vartheta)h \quad \text{in } \Omega.\end{aligned} \tag{24}$$

Multiplying (24) by $u$ and integrating over $\Omega$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2+\frac{\nu}{2}\|\mathbb{D}(u)\|_{L_2}^2\\ =-\int_\Omega u\cdot\nabla wu\,dx+\int_\Omega\alpha(\vartheta)hu\,dx+\int_\Omega(\alpha(\chi+\vartheta)-\alpha(\vartheta))fu\,dx\\ \le\varepsilon\|u\|_{H^1}^2+c(\varepsilon)\left[\|\nabla w\|_{L_3}^2\|u\|_{L_2}^2+\|h\|_{L_2}^2+\|h\|_{L_4}^2\left(\int_\Omega|\vartheta|^{4\sigma}dx\right)^{1/2}+\|\chi\|_{L_2}^2\|f\|_{L_3}^2\right],\end{aligned}$$

where we used the inequality $|\alpha(\chi+\vartheta)-\alpha(\vartheta)|\le a_3|\chi|$, which follows from the mean value theorem. Continuing and using the fact that $0<\sigma<1/8$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L_2}^2+\frac{\nu}{2}\|\mathbb{D}(u)\|_{L_2}^2\\ \le\varepsilon\|u\|_{H^1}^2+c(\varepsilon)(\|\nabla w\|_{L_3}^2\|u\|_{L_2}^2+\|h\|_{L_4}^2+\|h\|_{L_4}^2\|\vartheta\|_{L_2}+\|\chi\|_{L_2}^2\|f\|_{L_3}^2).\end{aligned} \tag{25}$$

Next, multiplying equation $(3)_3$ by $\chi$ and then integrating over $\Omega$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\chi\|_{L_2}^2+\varkappa\|\nabla\chi\|_{L_2}^2=-\int_\Omega w\cdot\nabla\chi\chi\,dx-\int_\Omega u\cdot\nabla\vartheta\chi\,dx+\nu\int_\Omega|\mathbb{D}(u)|^2\chi\,dx\\ +2\nu\int_\Omega\mathbb{D}(u):\mathbb{D}(w)\chi\,dx\equiv\sum_{i=1}^4 I_i.\end{aligned} \tag{26}$$

Estimate the terms $I_i$ on the right-hand side of (26). We have

$$I_1=0,\quad I_2\le\varepsilon\|\chi\|_{L_6}^2+c(\varepsilon)\|\nabla\vartheta\|_{L_2}^2\|u\|_{L_3}^2,\quad I_3\le c\|\nabla u\|_{L_3}^2\|\chi\|_{L_3}.$$

Applying the Gagliardo–Nirenberg interpolation inequality

$$\|v\|_{L_3} \leq c\|v\|_{H^1}^{1/2}\|v\|_{L_2}^{1/2} \quad \text{for } v \in \{\nabla u, \chi\} \tag{27}$$

we get

$$\begin{aligned} I_3 &\leq c\|\nabla u\|_{H^1}\|\nabla u\|_{L_2}\|\chi\|_{L_2}^{1/2}\|\chi\|_{H^1}^{1/2} \leq \varepsilon\|\nabla u\|_{H^1}^2 + c(\varepsilon)\|\nabla u\|_{L_2}^2\|\chi\|_{L_2}\|\chi\|_{H^1} \\ &\leq \varepsilon(\|\nabla u\|_{H^1}^2 + \|\chi\|_{H^1}^2) + c(\varepsilon)\|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2. \end{aligned}$$

Moreover,

$$I_4 \leq c\|\nabla u\|_{L_2}\|\nabla w\|_{L_3}\|\chi\|_{L_6} \leq \varepsilon\|\chi\|_{H^1}^2 + c(\varepsilon)\|\nabla u\|_{L_2}^2\|\nabla w\|_{L_3}^2.$$

Using the above estimates in (26) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\chi\|_{L_2}^2 + \varkappa\|\nabla\chi\|_{L_2}^2 \\ &\quad \leq \varepsilon(\|\nabla u\|_{H^1}^2 + \|\chi\|_{H^1}^2) + c(\varepsilon)(\|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2 + \|\nabla\vartheta\|_{L_2}^2\|u\|_{L_3}^2 + \|\nabla u\|_{L_2}^2\|\nabla w\|_{L_3}^2). \end{aligned} \tag{28}$$

Adding inequalities (25) and (28), assuming that $\varepsilon$ is sufficiently small and applying the Korn and Poincaré inequalities we get (23). □

**Lemma 15.** *Let the assumptions of Lemma* 14 *be satisfied. Then*

$$\begin{aligned} &\frac{d}{dt}\|\mathbb{D}(u)\|_{L_2}^2 + c(\|u\|_{H^2}^2 + \|\eta\|_{H^1}^2) \\ &\quad \leq c(\|u\|_{H^1}^6 + \|w\|_{W_4^1}^2\|u\|_{H^1}^2 + \|f\|_{L_\infty}^2\|\chi\|_{L_2}^2 + \|h\|_{L_4}^2\|\vartheta\|_{L_2} + \|h\|_{L_2}^2) \quad \text{in } (kT,(k+1)T),\ k \in \mathbb{N}_0, \end{aligned}$$

*where the constants c do not depend on k.*

*Proof.* Multiplying (24) by $-\operatorname{div}\mathbb{T}(u,\eta)$ and integrating over $\Omega$ yields

$$\begin{aligned} &\frac{\nu}{4}\frac{d}{dt}\|\mathbb{D}(u)\|_{L_2}^2 + \|\operatorname{div}\mathbb{T}(u,\eta)\|_{L_2}^2 \\ &\quad = \int_\Omega u\cdot\nabla u \operatorname{div}\mathbb{T}(u,\eta)\,dx + \int_\Omega u\cdot\nabla w \operatorname{div}\mathbb{T}(u,\eta)\,dx + \int_\Omega w\cdot\nabla w \operatorname{div}\mathbb{T}(u,\eta)\,dx \\ &\qquad - \int_\Omega \alpha(\vartheta)h \operatorname{div}\mathbb{T}(u,\eta)\,dx - \int_\Omega [\alpha(\chi+\vartheta) - \alpha(\vartheta)]f \operatorname{div}\mathbb{T}(u,\eta)\,dx. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{\nu}{4}\frac{d}{dt}\|\mathbb{D}(u)\|_{L_2}^2 + \|\operatorname{div}\mathbb{T}(u,\eta)\|_{L_2}^2 \\ &\quad \leq \varepsilon\|\operatorname{div}\mathbb{T}(u,\eta)\|_{L_2}^2 \\ &\qquad + c(\varepsilon)\bigg[\|u\cdot\nabla u\|_{L_2}^2 + \|w\cdot\nabla u\|_{L_2}^2 + \|u\cdot\nabla w\|_{L_2}^2 + \|h\|_{L_4}^2\bigg(\int_\Omega|\vartheta|^{4\sigma}\,dx\bigg)^{1/2} \\ &\qquad\qquad + \|h\|_{L_2}^2 + \|(\alpha(\chi+\vartheta) - \alpha(\vartheta))\chi f\|_{L_2}^2\bigg], \end{aligned}$$

where the constant $c$ do not depend on $k$.

To estimate $\|u\cdot\nabla u\|_{L_2}^2$ we use interpolation inequality (27), so we get

$$\|u\cdot\nabla u\|_{L_2}^2 \le c\|u\|_{L_6}^2\|\nabla u\|_{L_3}^2 \le c\|u\|_{H^1}^2\|\nabla u\|_{H^1}^2\|\nabla u\|_{L_2}^2 \le \varepsilon\|\nabla u\|_{H^1}^2 + c(\varepsilon)\|u\|_{H^1}^6.$$

Using the Hölder inequalities and the Sobolev imbedding to estimate the other terms on the right-hand side of (25), and then assuming that $\varepsilon$ is sufficiently small we obtain

$$\begin{aligned}\frac{d}{dt}\|\mathbb{D}(u)\|_{L_2}^2 + c\|\operatorname{div}\mathbb{T}(u,\eta)\|_{L_2}^2 \le c\big(&\|u\|_{H^1}^6 + \|w\|_{W_4^1}^2\|\nabla u\|_{L_2}^2 + \|\nabla w\|_{L_3}^2\|u\|_{H^1}^2\\ &+\|h\|_{L_4}^2\|\vartheta\|_{L_2}^{2\sigma} + \|h\|_{L_2}^2 + \|f\|_{L_\infty}^2\|\chi\|_{L_2}^2\big).\end{aligned}$$

Next, since by Lemma 10

$$\|u\|_{H^2} + \|\nabla\eta\|_{H^1} \le c\|\mathbb{T}(u,\eta)\|_{L_2},$$

the assertion of the lemma follows. $\square$

Lemmas 14 and 15 imply

**Lemma 16.** *Let the assumptions of Lemma 14 hold. Then*

$$\begin{aligned}&\frac{d}{dt}(\|u\|_{L_2}^2 + \|\mathbb{D}(u)\|_{L_2}^2 + \|\chi\|_{L_2}^2) + c(\|u\|_{H^2}^2 + \|\chi\|_{H^1}^2)\\ &\qquad\le c\big(\|u\|_{H^1}^6 + \|w\|_{W_4^1}^2\|u\|_{H^1}^2 + \|\nabla u\|_{L_2}^4\|\chi\|_{L_2}^2 + \|\nabla\vartheta\|_{L_2}^2\|u\|_{H^1}^2\\ &\qquad\quad + \|h\|_{L_4}^2 + \|h\|_{L_4}^2\|\vartheta\|_{L_2}^{2\sigma} + \|\chi\|_{L_2}^2\|f\|_{L_\infty}^2\big) \qquad \text{in } (kT,(k+1)T),\ k\in\mathbb{N}_0,\end{aligned} \tag{29}$$

*where the constants $c$ do not depend on $k$.*

Now, introduce notation:

$$\begin{aligned}X(t) &= \|Du(t)\|_{L_2}^2 + \|u(t)\|_{L_2}^2 + \|\chi(t)\|_{L_2}^2,\\ Y(t) &= \|u(t)\|_{H^2}^2 + \|\chi(t)\|_{H^1}^2,\\ A(t) &= \|w\|_{W_4^1}^2 + \|\nabla\hat{\vartheta}\|_{L_2}^2 + \|\nabla\bar{\theta}\|_{L_2}^2 + \|f(t)\|_{L_\infty},\\ G(t) &= \|h(t)\|_{L_4}^2, \quad B = \|\vartheta\|_{L_2}^{2\sigma}.\end{aligned}$$

**Lemma 17.** *Let the assumptions of Lemma* 14 *and Theorem* 4 *be satisfied. Assume that*

$$\sup_{k\in\mathbb{N}_0}\|f\|_{C([kT,(k+1)T];L_\infty)} \le \delta_1, \quad X(0)\le\gamma, \quad G(t)\le\delta_2\gamma \quad \text{for all } t\in\mathbb{R}_+.$$

*If the constants $\gamma$ and $\delta_i$ $(i=1,2)$ are sufficiently small then*

$$X(t)\le\gamma \quad \text{for all } t\in[kT,(k+1)T],\ k\in\mathbb{N}_0.$$

*Proof.* Inequality (29) yields

$$\frac{dX}{dt}+c_1X \le c_2(X^3+AX+G+BG), \qquad \text{for } t\in(kT,(k+1)T),\, k\in\mathbb{N}_0,$$

where $c_1, c_2 > 0$.

From Theorems 3 and 4 it follows that

$$A(t)\le c_3\delta_1 \qquad \text{for all } t\in(kT,(k+1)T],\, k\in\mathbb{N}_0,$$

where $c_3 > 0$.

Assume that $\delta_1$ is so small that $c_2c_3\delta_1 \le \frac{c_1}{2}$. Then we have

$$\frac{dX}{dt}+\frac{c_1}{2}X \le c_2(X^3+G+BG) \qquad \text{for all } t\in(kT,(k+1)T),\, k\in\mathbb{N}_0.$$

Next, assume that for some $k\in\mathbb{N}_0$

$$X(kT)\le\gamma.$$

Let

$$t_*=\inf\{t\in(kT,(k+1)T): X(t)>\gamma\}. \tag{30}$$

Then

$$\frac{dX}{dt}(t_*)+\frac{c_1}{2}\gamma \le c_2(\gamma^3+\delta_2\gamma+c_4\delta_2\gamma),$$

where $B\le c_4$, $c_4>0$. For $\delta_2$ and $\gamma$ so small that

$$c_2(\gamma^2+\delta_2+c_4\delta)\le\frac{c_1}{4}$$

we get

$$\frac{dX}{dt}(t_*)<0.$$

This is a contradiction with (30). Hence

$$X(t)\le\gamma \qquad \text{for } t\in[kT,(k+1)T].$$

This ends the proof. □

**Lemma 18.** *Let the assumption of Lemma* 17 *hold. Then*

$$\|u\|^2_{H^{2,1}(\Omega\times(kT,(k+1)T))}+\|\chi\|^2_{L_2(kT,(k+1)T;\,H^1)} \\ +\|\chi_t\|^2_{L_2(kT,(k+1)T;\,H^{-1})}+\|\nabla\eta\|^2_{L_2(kT,(k+1)T;\,L_2)}\le c(T)\gamma. \tag{31}$$

*Proof.* Integrating (29) with respect to time from $kT$ to $(k+1)T$ we get

$$\|u\|^2_{L_2(kT,(k+1)T;\,H^2)}+\|\chi\|^2_{L_2(kT,(k+1)T;\,H^1)}\le c(T)\gamma. \tag{32}$$

The other norms of (31) are estimated by using (32) and equations $(3)_{1,3}$. □

*The proofs of Theorems* 7 *and* 8. The assertions of Theorems 7 and 8 follow by the Galerkin approximations. We choose in $V$ a special basis which consists of eigenfunctions of the Stokes operator with the slip boundary conditions, and in $H_0^1(\Omega)$ we take a basis composed of eigenfunctions of the Laplace operator with the Dirichlet boundary condition. We repeat the proofs of Lemmas 14–18 (with slight modifications) to obtain inequalities (10) and (11) for the Galerkin approximations. Passing to the limit yields assertion of Theorem 7 and the existence of a strong-weak solution to problem (1).

In order to prove the uniqueness of a solution to problem (1) we have to show the uniqueness of a solution to problem (3). Let $(u_i,\chi_i,\eta_i)$, $i=1,2$, be two solutions of this problem and let $U=u_1-u_2$, $K=\chi_1-\chi_2$, $H=\eta_1-\eta_2$. Using the definition of the weak solution (see Definition 5) we get

$$\begin{aligned}
&\frac{d}{dt}\|U\|_{L_2}^2+\nu\|\mathbb{D}(U)\|_{L_2}^2\\
&\quad=-2\int_\Omega U\cdot\nabla u_1U\,dx-2\int_\Omega U\cdot\nabla wU\,dx+2\int_\Omega(\alpha(\vartheta+\chi_1)-\alpha(\vartheta+\chi_2))KfU\,dx\\
&\quad\le c(\|\nabla u_2\|_{L_2}+\|\nabla w\|_{L_2})\|U\|_{H^1}^2+\varepsilon\|U\|_{H^1}^2+C(\varepsilon)\|K\|_{L_2}^2\|f\|_{L_\infty}^2,
\end{aligned}\tag{33}$$

and

$$\begin{aligned}
&\frac{d}{dt}\|K\|_{L_2}^2+2\varkappa\|\nabla K\|_{L_2}^2\\
&\quad=-2\int_\Omega U\cdot\nabla\chi_1K\,dx-2\int_\Omega U\cdot\nabla\vartheta K\,dx+2\nu\int_\Omega|\mathbb{D}(U)|^2K\,dx\\
&\qquad+4\nu\int_\Omega\mathbb{D}(U):\mathbb{D}(u_2)K\,dx+4\nu\int_\Omega\mathbb{D}(U):\mathbb{D}(w)K\,dx\\
&\quad\le c\big[(\|\nabla\chi_1\|_{L_2}^2+\|\nabla\vartheta\|_{L_2}^2+\|\nabla u_2\|_{H^1}^2+\|\nabla w\|_{H^1}^2)\|U\|_{H^1}^2+\|U\|_{H^1}^4\|K\|_{L_2}^2\big]+\varepsilon\|K\|_{H^1}^2.
\end{aligned}\tag{34}$$

Now, adding (33) and (34), and using Theorems 3, 4 and 7 we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|U\|_{L_2}^2+\|K\|_{L_2}^2)+c(\|U\|_{H^1}^2+\|K\|_{H^1}^2)\\
&\quad\le c(\gamma+\delta_1)(\|U\|_{H^1}^2+\|K\|_{L_2}^2)+\varepsilon(\|U\|_{H^2}^2+\|K\|_{H^1}^2)\\
&\qquad+c(\|\nabla\chi_1\|_{L_2}^2+\|\nabla u_2\|_{H^1}^2)\|U\|_{H^1}^2.
\end{aligned}\tag{35}$$

Assuming that $\gamma$, $\delta_1$ and $\varepsilon$ are sufficiently small inequality (35) we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|U\|_{L_2}^2+\|K\|_{L_2}^2)+c(\|U\|_{H^1}^2+\|K\|_{H^1}^2)\\
&\qquad\le c(\|\nabla\chi_1\|_{L_2}^2+\|\nabla u_2\|_{H^1}^2)\|U\|_{H^1}^2+\varepsilon\|U\|_{H^2}^2.
\end{aligned}\tag{36}$$

In order to derive an estimate for the norm $\|U\|_{H^2}^2$ we consider the equation

$$\begin{aligned}
&U_t-\operatorname{div}\mathbb{T}(U,H)+U\cdot\nabla U\\
&\qquad=-U\cdot\nabla u_2-u_2\cdot\nabla U-w\cdot\nabla U-U\cdot\nabla w+\alpha'Kf\quad\text{in }\Omega\times\mathbb{R}_+.
\end{aligned}$$

Multiplying the above equation by $-\operatorname{div}\mathbb{T}(U,H)$ we have

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\mathbb{D}(U)\|_{L_2}^2+\|\operatorname{div}\mathbb{T}(U,H)\|_{L_2}^2 \\ \le \varepsilon\|\operatorname{div}\mathbb{T}(U,H)\|_{L_2}^2+c(\varepsilon)\big[(\|U\|_{L_6}^2+\|u_2\|_{L_6}^2\|w\|_{L_6}^2)\|\nabla U\|_{L_3}^2 \\ +(\|\nabla w\|_{L_3}^2+\|\nabla u_2\|_{L_3}^2)\|U\|_{L_6}^2+\|K\|_{L_2}^2\|f\|_{L_\infty}^2\big].\end{aligned}$$

Hence for sufficiently small $\varepsilon$

$$\frac{d}{dt}\|\mathbb{D}(U)\|_{L_2}^2+c\|U\|_{H^2}^2\le c(\gamma+\delta_1)\|U\|_{H^2}^2+c\delta_1\|K\|_{L_2}^2+c\|\nabla u_2\|_{H^1}^2\|U\|_{H^1}^2. \tag{37}$$

Adding (36) and (37) we get

$$\frac{d}{dt}(\|\mathbb{D}(U)\|_{L_2}^2+\|U\|_{L_2}^2+\|K\|_{L_2}^2)+c(\|U\|_{H^2}^2+\|K\|_{H^1}^2)\le c(\|\nabla\chi_1\|_{L_2}^2+\|\nabla u_2\|_{H^1}^2)\|U\|_{H^1}^2,$$

for sufficiently small $\varepsilon$, $\gamma$ and $\delta_1$, which implies that

$$\|U(t)\|_{H^1}+\|K(t)\|_{H^1}=0.$$

This completes the proof. □

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# MISIUREWICZ PARAMETERS FOR WEIERSTRASS ELLIPTIC FUNCTIONS BASED ON TRIANGLE AND SQUARE LATTICES

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**Abstract:** For two families of Weierstrass elliptic functions – based on triangular or square lattices – we prove that the set of Misiurewicz parameters has the Lebesgue measure zero in $\mathbb{C}$.
**Keywords:** meromorphic transcendental functions, Weierstrass elliptic functions, Misiurewicz condition
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## 1. INTRODUCTION

We consider Weierstrass elliptic functions based on the lattice

$$\Lambda = \{m\lambda_1 + n\lambda_2 \colon m,n \in \mathbb{Z}\} =: [\lambda_1, \lambda_2]\ ,\ \lambda_2/\lambda_1 \notin \mathbb{R},$$

given by the formula

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

It is a wide class of meromorphic functions, periodic with respect to $\Lambda$ and of order two. We refer the reader to [5, 6] for a general description of dynamical and measure-theoretic properties of $\wp_\Lambda$ depending on the lattice $\Lambda$. Some results on specific parametrized families of Weierstrass elliptic functions can be found there as well. For an introduction to the theory of iterating complex functions see e.g. [3].

Even fixing the type of the lattice $\Lambda$, i.e., the shape $\tau = \lambda_2/\lambda_1$ of the corresponding period parallelogram of $\wp_\Lambda$, we still obtain an incredible richness of dynamical behaviour and properties of Weierstrass functions. We are particularly interested in two families of functions: those based on triangular lattices, i.e., satisfying $e^{2\pi i/3}\Lambda = \Lambda$, and those based on square lattices, that is lattices such that $i\Lambda = \Lambda$. Let us specify the families $\mathcal{W}_t$ and $\mathcal{W}_s$ we are interested in.

The family $\mathcal{W}_t$ consists of all Weierstrass elliptic functions based on triangular lattices. Formally,

$$\mathcal{W}_t = \left\{ f_\lambda := \wp_{\Lambda_\lambda} : \mathbb{C} \to \overline{\mathbb{C}}, \text{ where } \Lambda_\lambda = [\lambda, e^{2\pi i/3}\lambda],\ \lambda \in \mathbb{C} \setminus \{0\} \right\}.$$

All Weierstrass elliptic functions based on square lattices are members of the family $\mathcal{W}_s$, i.e.,

$$\mathcal{W}_s = \left\{ f_\lambda := \wp_{\Lambda_\lambda} : \mathbb{C} \to \overline{\mathbb{C}}, \text{ where } \Lambda_\lambda = [\lambda, \lambda i],\ \lambda \in \mathbb{C} \setminus \{0\} \right\}.$$

Since most of the considerations are the same for both families, we are not to restrictive about the notation. We will point out the differences when necessary.

The dynamics of these functions is fairly rigid because of the close relationship between trajectories of critical values. Therefore, there are only a couple of possible structures of the Fatou set that may occur – we will list them in the next section (Lemma 5 and Lemma 6). In this paper, we will show that one of the cases, i.e. when $f_\lambda$ satisfies the so-called Misiurewicz condition, appears very rarely.

The notion of Misiurewicz maps derives from the paper [9] by M. Misiurewicz, where the author studied, among other topics, the real quadratic family $g_a(x) = 1 - ax^2$ in the case when $g_a$ is non-hyperbolic and the critical point 0 is non-recurrent. We refer the reader to [1] for a nice discussion concerning various definitions of the Misiurewicz condition in the complex case. For the considered families of Weierstrass elliptic function we introduce the following definition.

**Definition 1.** *A function $f_\lambda$ from the family $\mathcal{W}_t$ or $\mathcal{W}_s$ satisfies the Misiurewicz condition (equivalently $\lambda$ is a Misiurewicz parameter) if all singular values of $f_\lambda$ belong to the Julia set and the set $\mathcal{P}(f_\lambda) \cap \mathbb{C}$ (the finite part of the postsingular set) is bounded and disjoint from the set* $\mathrm{Crit}(f_\lambda)$ *of the critical points of $f_\lambda$.*

In other words, every singular value of $f_\lambda$ is either a prepole or has a bounded trajectory staying at a positive distance from the set of critical points $\mathrm{Crit}(f_\lambda)$. This may seem more restrictive than the definition introduced by Graczyk, Kotus and Świątek in [4], as we demand that all singular values lie in the Julia set, but after analysis of dynamics of functions from the considered families it will become clear that the above definition is natural in this case. Note also that the definition includes the case (sometimes referred as *pure Misiurewicz*) when all singular values are preperiodic.

It was proved by M. Aspenberg in [1] that the set of Misiurewicz maps has the Lebesgue measure zero in the space of rational functions of any fixed degree. Next, this result was extended in [2] to the exponential family, which is one-dimensional space of entire transcendental maps. In this paper, we generalize these results and prove the following theorem.

**Theorem 2.** *For the families $\mathcal{W}_t$ and $\mathcal{W}_s$, the set of Misiurewicz parameters has the Lebesgue measure zero in $\mathbb{C}$.*

We will prove this result in two steps. First, we deal with parameters to which we can apply similar technique as in [1, 2] and show that the following theorem is true:

**Theorem 3.** *For the families $\mathcal{W}_t$ and $\mathcal{W}_s$, the set of parameters $\lambda$ for which there exists in the Julia set $J(f_\lambda)$ a critical value which is not a prepole and has a bounded trajectory not accumulating on the critical set $\mathrm{Crit}(f_\lambda)$ has the Lebesgue measure zero in $\mathbb{C}$.*

Because of the close relationship between all critical trajectories in the considered families, the assumptions of Theorem 3 imply, in particular, that all critical values of $f_\lambda$ (except for the pole 0 in the case of a square lattice) are not prepoles and have bounded trajectories in $J(f_\lambda)$ separated from $\mathrm{Crit}(f_\lambda)$, hence $f_\lambda$ is a special case of a Misiurewicz map.

However, in order to deal with all Misiurewicz parameters, we need to consider one more case, i.e. when all critical values of $f_\lambda$ are prepoles. Therefore, at the end of the article we will prove the following lemma.

**Lemma 4.** *For the families $\mathcal{W}_t$ and $\mathcal{W}_s$, the set of parameters $\lambda$ for which all critical values of $f_\lambda$ are prepoles is countable.*

Note that Theorem 3 and Lemma 4 imply the main result of the paper, i.e. Theorem 2, since elliptic functions have no asymptotic values.

The proof of Theorem 3 in general follows the Aspenberg's approach from [1], repeated in [2] with some changes for the exponential family. Note, however, that in our case we face new difficulties: we have to deal not only with infinite degree of maps and essential singularity at $\infty$, but also with prepoles which become essential singularities in $\mathbb{C}$ for iterates of considered functions. That is why we have to be sure that we can stay away from poles and essential singularities in order to proceed with calculations. Some minor but crucial changes had to be done especially in the section 3.1, where we prove existence of a holomorphic motion and the so-called transversality condition and for measure estimates in a big scale in the section 3.4 (see Lemma 19).

Lemma 4 is proved at the end of the paper. We describe the condition that all critical values are prepoles by an analytic equation depending on a countable number of parameters (this is possible because of the close relationship between critical values of considered functions). Next, using postsingular stability, $\lambda$-lemma and the nonexistence of invariant line fields (see [10, Theorem 1.1]), we show that roots of the equation are isolated, hence there are only countably many parameters for which all critical values are prepoles.

# 2. DYNAMICS OF FUNCTIONS FROM FAMILIES $\mathcal{W}_T$ AND $\mathcal{W}_S$

Recall that an elliptic function has no asymptotic values, so the postsingular set $\mathcal{P}(f_\lambda)$ is the closure of the critical trajectories. Moreover, the Fatou set of any Weierstrass elliptic function contains neither wandering domains, nor Baker domains, nor Herman rings (see [6, Lemma 5.2, Theorem 5.4]).

Take any function $f_\lambda \in \mathcal{W}_t$. It has three critical values $e_1$, $e_2$ and $e_3$, all with the same modulus and forming the angle $2\pi i/3$ with each other on the complex plane, i.e., $e_2 = \mathrm{e}^{2\pi i/3} e_1$ and $e_3 = \mathrm{e}^{4\pi i/3} e_1$. Recall that the triangular lattice is invariant under the rotation by the angle $2\pi i/3$, thus the homogenity properties (cf. (3) in [6]) give that the same relationship holds for every iterate of critical values, i.e. $f_\lambda^n(e_2) = \mathrm{e}^{2\pi i/3} f_\lambda^n(e_1)$ and $f_\lambda^n(e_3) = \mathrm{e}^{4\pi i/3} f_\lambda^n(e_1)$. Moreover, for any $n \geq 0$, the derivative $f_\lambda'(f_\lambda^n(e_i))$ is the same for $i = 1,2,3$. As a consequence we obtain the following result (see [6, Proposition 5.3]):

**Lemma 5.** *For any function $f_\lambda \in \mathcal{W}_t$, one of the following cases occurs:*

1. $J(f_\lambda) = \overline{\mathbb{C}}$.
2. *For some period n and multiplier $0 \leq \beta \leq 1$, there exist exactly three (super) attracting or parabolic periodic cycles in $F(f_\lambda)$ of period n with multiplier $\beta$.*
3. *There exists exactly one (super) attracting or parabolic periodic cycle in $F(f_\lambda)$ which contains all three critical values.*
4. *The only Fatou cycles are Siegel discs.*

Since the dynamics of all three critical values is basically the same, it is enough to know one of them to determine the other two. In particular, if the assumptions of Theorem 3 are satisfied, then every $e_i$ is not a prepole and has a bounded trajectory in $J(f_\lambda)$, separated from $\mathrm{Crit}(f_\lambda)$. On the other hand, if one critical value is a prepole, so are the other two.

In case of square lattices, take some $f_\lambda \in \mathcal{W}_s$. We have the following critical values: $e_1$, $e_2 = -e_1$ and $e_3 = 0$, which is a pole of $f_\lambda$, so the situation is even more rigid than before. By the definition, $f_\lambda$ is even, so $e_1$ and $e_2$ share the same trajectory, which actually determines the dynamics of $f_\lambda$ since $e_3$ is always a pole. Thus, there are only three possibilities (see [6, Proposition 5.4]).

**Lemma 6.** *For any function $f_\lambda \in \mathcal{W}_s$, one of the following cases occurs:*

1. $J(f_\lambda) = \overline{\mathbb{C}}$.
2. *There exists exactly one (super) attracting or parabolic periodic cycle in $F(f_\lambda)$.*
3. *The only Fatou cycles are Siegel discs.*

Now, if the assumptions of Theorem 3 are satisfied, then all critical values are in $J(f_\lambda)$. Moreover, the trajectory of $e_1$ and $e_2$, which are not prepoles in this case, is bounded and separated from $\mathrm{Crit}(f_\lambda)$. And just as for triangle lattices, if $e_1$ or $e_2$ is a prepole, then all critical values of $f_\lambda$ are prepoles.

As we mentioned at the beginning there are various definitions of the Misiurewicz condition in the complex case. One of the classical definitions, sometimes referred to as *pure Misiurewicz*, demands that every singular value is preperiodic, i.e., is eventually mapped onto a repelling periodic cycle in the Julia set. This condition, however, is very restrictive and we usually introduce more general definitions very often depending on the family of functions under consideration. In our case Definition 1 was inspired by the close relationship between critical trajectories of functions from families $\mathcal{W}_t$ and $\mathcal{W}_s$.

# 3. PROOF OF THEOREM 3

Denote by $\mathcal{M}$ the set of parameters $\lambda$ satisfying the assumptions of Theorem 3 and by $e_\lambda \in J(f_\lambda)$ the critical value of $f_\lambda$ (which is not a prepole) with bounded trajectory not accumulating on $\mathrm{Crit}(f_\lambda)$. It follows that for every $\lambda \in \mathcal{M}$ we can find some $\delta > 0$ such that

$$\overline{O_\lambda(e_\lambda)} \cap \Big(B\big(\mathrm{Crit}(f_\lambda), \delta\big) \cup B\big(\infty, \delta\big)\Big) = \emptyset, \tag{1}$$

where $O_\lambda(e_\lambda) = \bigcup_{n\geq 1} f_\lambda^n(e_\lambda)$ is the forward trajectory of the critical value $e_\lambda$ and balls are taken with respect to the spherical metric. The set of parameters for which (1) holds for any critical value $e_\lambda \in J(f_\lambda)$ of $f_\lambda$ will be denoted by $\mathcal{M}_\delta$. Note that

$$\mathcal{M} = \bigcup_{n\geq 1} \mathcal{M}_{1/n} \quad \text{and} \quad \delta_1 < \delta_2 \Rightarrow \mathcal{M}_{\delta_1} \supset \mathcal{M}_{\delta_2}.$$

Similarly to the case of the exponential family (cf. [2]), we will show, following Aspenberg's idea in [1], that parameters from $\mathcal{M}_\delta$ are rare in any neighbourhood of $\lambda_0 \in \mathcal{M}$.

**Theorem 7.** *For families $\mathcal{W}_t$ and $\mathcal{W}_s$, if $\lambda_0 \in \mathcal{M}$, then for every $\delta > 0$ the set $\mathcal{M}_\delta$ has the Lebesgue density strictly smaller than 1 at $\lambda_0$.*

Obviously, Theorem 7 implies that $\mu(\mathcal{M}_\delta) = 0$ for every $\delta > 0$, where $\mu$ is the Lebesgue measure on $\overline{\mathbb{C}}$. Hence,

$$\mu(\mathcal{M}) \leq \sum_{n\geq 1} \mu(\mathcal{M}_{1/n}) = 0,$$

which is exactly the conclusion of Theorem 3.

In order to prove Theorem 7, we will focus on parameter $\lambda_0 \in \mathcal{M}$ and its neighbourhood $B(\lambda_0, r)$ in the parameter plane. We will see how the assumptions on the critical value $e_{\lambda_0}$ and dynamical properties of families $\mathcal{W}_t$ and $\mathcal{W}_s$ imply exponential expansion on $\mathcal{H}$, the closure of the forward trajectory of $e_{\lambda_0}$ under $f_{\lambda_0}$. This leads to the existence of a holomorphic motion $h: \mathcal{H} \times B(\lambda_0, r) \to \mathbb{C}$ conjugating the dynamics of $f_{\lambda_0}$ and nearby maps $f_\lambda$, $\lambda \in B(\lambda_0, r)$, on a neighbourhood of $\mathcal{H}$. Next, we will use the expansion property and the absence of line fields for Misiurewicz elliptic maps to derive nice distortion properties binding space and parameter derivatives in a small scale. This allows us to control the growth of the parameter ball $B(\lambda_0, r)$ to a big scale, where in turn we can estimate the measure of those parameters which cannot belong to $\mathcal{M}_\delta \subset \mathcal{M}$.

## 3.1. HOLOMORPHIC MOTION

Take a parameter $\lambda_0 \in \mathcal{M}$ for either one of those two families. As we have just seen, all critical values of $f_{\lambda_0}$ are in the Julia set $J(f_{\lambda_0})$. Recall that the Fatou set $F(f_{\lambda_0})$ has neither wandering domains, nor Baker domains, nor Herman rings. Moreover, $f_{\lambda_0}$ is expanding on the closure of the critical trajectory, hence the close relationship between trajectories of all critical values excludes the existence of Siegel discs. We conclude that the Fatou set must be empty, thus $J(f_{\lambda_0}) = \overline{\mathbb{C}}$. Now, pick one of the critical values in $J(f_{\lambda_0})$ which is not a pole and denote it by $e_{\lambda_0}$. Here and in the following sections we use the spherical metric and derivatives, unless otherwise stated.

Consider $\mathcal{H} = \overline{O_{\lambda_0}(e_{\lambda_0})}$, the closure of the forward trajectory of $e_{\lambda_0}$ under $f_{\lambda_0}$. It is compact, forward invariant, contains neither critical nor parabolic points. Hence, by Theorem 1.2 in [10] (compare also with [4, Theorem 1]), $\mathcal{H}$ is a hyperbolic set, i.e., there exist real constants $C > 0$ and $a > 1$ such that

$$|(f_{\lambda_0}^n)'(z)| \geq Ca^n \quad \text{for all } z \in \mathcal{H} \text{ and } n \geq 1.$$

Now, look at the nearby maps $f_\lambda$, $\lambda \in B(\lambda_0, r)$, either in $\mathcal{W}_t$ or in $\mathcal{W}_s$. We will follow the proof of [8, Theorem III.1.6] locally in a neighbourhood of the hyperbolic set $\mathcal{H}$ to show that if $r > 0$ is sufficiently small, there exists a holomorphic motion

$$h\colon \mathcal{H} \times B(\lambda_0, r) \to \mathbb{C}$$

such that $h_{\lambda_0} = \mathrm{id}$, the map $h_\lambda := h(\cdot, \lambda)\colon \mathcal{H} \to \mathcal{H}_\lambda$ is quasiconformal for each $\lambda \in B(\lambda_0, r)$ and $h(z, \cdot)\colon B(\lambda_0, r) \to \mathbb{C}$ is holomorphic at every $z \in \mathcal{H}$. Moreover, it respects the dynamics, i.e.,

$$h_\lambda \circ f_{\lambda_0} = f_\lambda \circ h_\lambda \quad \text{on } \mathcal{H}.$$

First, notice that $\mathcal{H}$ contains no prepoles of $f_{\lambda_0}$. Fix an $N \in \mathbb{N}$ such that

$$\forall\, z \in \mathcal{H}, \quad |(f_{\lambda_0}^N)'(z)| \geq 2\tilde{a}$$

for some constant $\tilde{a} \gg 1$. Now, take a neighbourhood $\mathcal{N}$ of $\mathcal{H}$ such that even in a bigger neighbourhood $\mathcal{N}_\varepsilon = B(\mathcal{N}, \varepsilon)$, for some $\varepsilon > 0$, there are neither critical points of $f_{\lambda_0}$, nor prepoles of $f_{\lambda_0}$ of orders $1, 2, \ldots, N$.

Now, we want to choose a sufficiently small radius $r > 1$ in the parameter space. We do it in two steps, decreasing $\mathcal{N}$ if necessary, so that the following two conditions are satisfied:

1. $\forall\, \lambda \in B(\lambda_0, r), \mathcal{N}$ contains neither critical points nor prepoles of $f_\lambda$ of orders $1, 2, \ldots, N$.
2. $\forall\, \lambda \in B(\lambda_0, r),\ \forall\, z \in \mathcal{N}, \quad |(f_\lambda^N)'(z)| \geq \tilde{a} \gg 1.$

It is possible since the critical points and poles depend analytically on the parameter $\lambda$ and the derivative $(f_\lambda^N)'(z)$ changes continuously with $\lambda$.

The choice of $r > 0$ guarantees the expanding property for all functions $f_\lambda$, $\lambda \in B(\lambda_0, r)$, where the constants $C > 0$ and $a > 1$ might have changed.

**Lemma 8.** *There exist numbers $C > 0$, $a > 1$ and $r > 0$ such that whenever $f_\lambda^j(z) \in \mathcal{N}$ for $j = 0, \ldots, k$ and $\lambda \in B(\lambda_0, r)$, then*

$$|(f_\lambda^k)'(z)| \geq Ca^k.$$

The next step is to introduce an appropriate adapted metric defined for $z \in \mathcal{N}$ as follows:

$$d(z) = \frac{1}{N} \sum_{n=0}^{N-1} |(f_{\lambda_0}^n)'(z)|.$$

Choosing $\mathcal{N}$ carefully we get $d(z) \leq C_1$ for all $z \in \mathcal{N}$. Additionally, we can modify $C_1$, so that the estimate remains valid for every function $f_\lambda$, $\lambda \in B(\lambda_0, r)$, decreasing $r$ if necessary.

Let us compute derivative $|f'|_d$ of the function $f := f_{\lambda_0}$ with respect to the adapted metric for $z \in \mathcal{N}$.

$$\begin{aligned} |f'(z)|_d =& |f'(z)| \frac{d(f(z))}{d(z)} = \frac{|f'(z)| \frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(f(z))|}{\frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(z)|} = \frac{\frac{1}{N} \sum_{n=0}^{N-1} |(f^{n+1})'(z)|}{\frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(z)|} \\ =& 1 + \frac{\frac{1}{N}(|(f^N)'(z)| - 1)}{\frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(z)|} \geq 1 + \frac{\tilde{a} - 1}{NC_1} > 1, \end{aligned}$$

hence $|(f_{\lambda_0})'|_d \geq const > 1$ on $\mathcal{N}$.

Take a nearby function $g := f_\lambda$, where $\lambda \in B(\lambda_0, r)$ for sufficiently small $r > 0$, and $z \in \mathcal{N}$.

$$|g'(z)|_d = |g'(z)| \frac{d(g(z))}{d(z)} = \frac{|g'(z)| \frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(g(z))|}{\frac{1}{N} \sum_{n=0}^{N-1} |(f^n)'(z)|} = \frac{\frac{1}{N} \sum_{n=0}^{N-1} |(f^n \circ g)'(z)|}{\frac{1}{N} \sum_{n=0}^{N-1} |(f^{n+1})'(z)|} |f'(z)|_d.$$

Since $|(f_{\lambda_0})'(z)|_d \geq const > 1$ on $\mathcal{N}$, if follows that if $r > 0$ is sufficiently small (decreasing $\mathcal{N}$ if necessary), then for any $\lambda \in B(\lambda_0, r)$

$$|(f_\lambda)'|_d \geq \tilde{C} > 1 \quad \text{on} \quad \mathcal{N}.$$

This is a consequence of the form of the derivative with respect to the adapted metric as we consider only finitely many iterates, there are no prepoles of $f_\lambda$ of orders $1, 2, \ldots, N$ in $\mathcal{N}$ and values of functions and iterates (which are holomorphic, bounded and equicontinuous on $\mathcal{N}$) depend continuously on $\lambda$.

We proceed exactly as in [8]. Let $\varepsilon > 0$ be such that for every $z \in \mathcal{H}$, $B(z, \varepsilon)_d \subset \mathcal{N}$ (the ball with respect to the adapted metric). If $r > 0$ is sufficiently small, then for every $\lambda \in B(\lambda_0, r)$ we have $f_\lambda(B(z, \varepsilon)_d) \supset B(f_{\lambda_0}(z), \varepsilon)_d$. Hence, for every $n \in \mathbb{N}$ and $z \in \mathcal{H}$, the set

$$W_{\lambda,n} = \left\{ w : f_\lambda^k(w) \in B\left(f_{\lambda_0}^k(z), \varepsilon\right)_d \text{ for } k = 0, 1, \ldots, n \right\}$$

is nonempty and its diameter does not exceed $2\varepsilon\tilde{C}^{-n}$. Therefore, there exists a unique point $h_\lambda(z)$ such that $f_\lambda^n(h_\lambda(z)) \in B(f_{\lambda_0}^n(z), \varepsilon)_d$ for all $n \in \mathbb{N}$. We immediately get $h_\lambda(f_{\lambda_0}(z)) = f_\lambda(h_\lambda(z))$. Moreover, $h_\lambda$ is continuous and injective.

Since the holomorphic motion $h \colon \mathcal{H} \times B(\lambda_0, r) \to \mathbb{C}$ respects the dynamics and $f_{\lambda_0}(\mathcal{H}) \subset \mathcal{H}$, we get

$$f_\lambda(h_\lambda(\mathcal{H})) = h_\lambda(f_{\lambda_0}(\mathcal{H})) \subset h_\lambda(\mathcal{H}),$$

thus the set $\mathcal{H}_\lambda := h_\lambda(\mathcal{H})$ is $f_\lambda$-invariant and by the Lemma 8, it is a hyperbolic set for $f_\lambda$.

Now, we want to obtain the so-called transversality condition (cf. [1]), which says that the critical value $e_\lambda$ of $f_\lambda$ cannot follow the holomorphic motion $h_\lambda(e_{\lambda_0})$ of the critical value of $f_{\lambda_0}$ in the whole parameter ball $B(\lambda_0, r)$. In the triangular case, it follows from the non-existence of invariant line-fields for Misiurewicz maps proved by Graczyk, Kotus and Świątek in [4, Theorem 2]. For the case of square lattices we refer the reader to the more general result [10, Theorem 1.1]. For convenience, we will use notation analogous to [1].

Recall that there is a strong relationship between the trajectories of critical values of functions in both families $\mathcal{W}_t$ and $\mathcal{W}_s$, in particular the trajectory of $e_\lambda$ determines the dynamics of $f_\lambda$. Consider a holomorphic function $x : B(\lambda_0, r) \to \mathbb{C}$ given by

$$x(\lambda) = e_\lambda - h_\lambda(e_{\lambda_0})$$

which is exactly the difference between the critical value of $f_\lambda$ and the holomorphic motion of the critical value of the starting map $f_{\lambda_0}$ (we assume that the radius of the parameter ball is so small that there is only one critical value of $f_\lambda$ close to $e_{\lambda_0}$). Note that $h_\lambda(e_{\lambda_0})$ always belongs to the hyperbolic set $\mathcal{H}_\lambda$. Obviously $x(\lambda_0) = 0$. Our aim is to show that $\lambda_0$ is an isolated zero of $x$.

**Lemma 9.** *The function $x$ is not identically zero in any ball $B(\lambda_0, r)$ in the parameter plane.*

*Proof.* Suppose that $x(\lambda) \equiv 0$ on some ball $B(\lambda_0, r)$, which means that for any $\lambda$ close to $\lambda_0$ the trajectory of the critical value $e_\lambda$ stays in the appropriate hyperbolic set $\mathcal{H}_\lambda$. It follows that the trajectories of all critical values of $f_\lambda$, except for the pole $e_3$ in the case of square lattice, lie in some hyperbolic set. Thus, the parameter $\lambda_0$ is postsingularly stable, since trajectories of all critical values of $f_\lambda$ behave the same way for all parameters $\lambda$ close to $\lambda_0$. We can, therefore, extend $h_\lambda$ to a quasiconformal conjugacy on the consecutive preimages of $e_\lambda$ and next, by the $\lambda$-Lemma (cf. [7, $\lambda$-Lemma]), to a quasiconformal conjugacy on the whole Julia set $J(f_{\lambda_0}) = \overline{\mathbb{C}}$ between $f_{\lambda_0}$ and $f_\lambda$ for any $\lambda \in B(\lambda_0, r)$. In this case, however, there would be an $f_{\lambda_0}$-invariant line field on $J(f_{\lambda_0})$ which cannot exist by [10, Theorem 1.1] (cf. [4, Theorem 2]). $\square$

Therefore we have

$$x(\lambda) = \alpha_K(\lambda - \lambda_0)^K + \alpha_{K+1}(\lambda - \lambda_0)^{K+1} + \dots \tag{2}$$

for some $K \geq 1$ and $\alpha_K \neq 0$. This property will be crucial to obtain distortion estimates in the next section.

## 3.2. DISTORTION ESTIMATES

In this section, we derive distortion estimates based on the expansion property near the hyperbolic set $\mathcal{H}$. It is rather technical and mainly follows the analogous proofs in [1] and [2]. We decided however to keep it in a very detailed form for the convenience of the reader and also because of are minor but crucial differences.

Recall that we have chosen the neighbourhood $\mathcal{N}$ of the hyperbolic set $\mathcal{H}$ and $r > 0$, so that for all functions $f_\lambda$, $\lambda \in B(\lambda_0, r)$, we have the expansion property stated in Lemma 8. Assume, moreover, that $\mathcal{N}$ is closed, bounded (hence compact in $\mathbb{C}$) and for some $\delta > 0$

$$\mathcal{N} \cap \big(B\big(\mathrm{Crit}(f_\lambda), \delta\big) \cup B\big(\infty, \delta\big)\big) = \emptyset.$$

From now on, when we take some $\delta' > 0$ for which $\{z : \mathrm{dist}(z, \mathcal{H}) \leq 11\delta'\} \subset \mathcal{N}$, we always assume $r > 0$ to be so small that $\{z : \mathrm{dist}(z, \mathcal{H}_\lambda) \leq 10\delta'\} \subset \mathcal{N}$ for each $\lambda \in B(\lambda_0, r)$. This means that $\mathcal{H}_\lambda$, the hyperbolic set for $f_\lambda$, is well inside $\mathcal{N}$.

The neighbourhood $\mathcal{N}$ was chosen so that for some $N \geq 1$, $\tilde{a} > 1$ and for all $z \in \mathcal{N}$, $\lambda \in B(\lambda_0, r)$, we have $|(f_\lambda^N)'(z)| \geq \tilde{a}$. Thus, for every $z \in \mathcal{N}$ we can find a number $r(z) > 0$ such that

$$|f_\lambda^N(z) - f_\lambda^N(w)| \geq \tilde{a}|z - w| \tag{3}$$

for all $w \in \mathcal{N}$ with $|z - w| \leq r(z)$ (decreasing slightly $\tilde{a} > 1$ if necessarily). Since $\mathcal{N}$ is compact and $r(z)$ changes continuously, we can find a universal $\tilde{r} > 1$ such that (3) holds for every $z, w \in \mathcal{N}$ with $|z - w| \leq \tilde{r}$. This implies exponential expansion in a small scale.

**Lemma 10.** *There are constants $\tilde{\delta}, C > 0$ and $a > 1$ such that for every $\lambda \in B(\lambda_0, r)$ and for every $z, w \in \mathcal{N}$, if $f_\lambda^j(z), f_\lambda^j(w) \in \mathcal{N}$ and $|f_\lambda^j(z) - f_\lambda^j(w)| \le \tilde{\delta}$ for $j = 0, \ldots, k$, then*

$$|f_\lambda^k(z) - f_\lambda^k(w)| \ge Ca^k|z-w|.$$

*Proof.* Every integer $k$ can be written in the form $k = pN + q$, where $q \le N-1$. For some $\tilde{C}, \tilde{\delta} > 0$ we can estimate for all $\lambda \in B(\lambda_0, r)$

$$|f_\lambda(z) - f_\lambda(w)| \ge \tilde{C}|z-w| \ \text{ for all } z, w \in \mathcal{N} \ \text{ with } |z-w| \le \tilde{\delta}.$$

If we take $z, w \in \mathcal{N}$ satisfying assumptions of the lemma, then

$$|f_\lambda^k(z) - f_\lambda^k(w)| \ge \tilde{a}^p|f_\lambda^q(z) - f_\lambda^q(w)| \ge \tilde{a}^p\tilde{C}^q|z-w| \ge a^kC|z-w|$$

for $a = \tilde{a}^{\frac{1}{N}}$ and some $C > 0$. □

We will use the expansion property in the following distortion estimates to show that in a small scale parameter and space derivatives are comparable. For $\lambda \in B(\lambda_0, r)$ and $n \ge 0$, put

$$\xi_n(\lambda) = f_\lambda^n(e_\lambda) \quad \text{and} \quad \mu_n(\lambda) = f_\lambda^n(h_\lambda(e_{\lambda_0})) = h_\lambda(f_{\lambda_0}^n(e_{\lambda_0})).$$

Then $\xi_n(\lambda)$ is the forward orbit of the critical value for $f_\lambda$, while $\mu_n(\lambda)$ is the holomorphic motion of the critical orbit for $f_{\lambda_0}$, hence $\mu_n(\lambda) \in \mathcal{H}_\lambda$. In particular, $x(\lambda) = \xi_0(\lambda) - \mu_0(\lambda)$.

The following lemma will be used several times in our distortion estimates (see [1] for references).

**Lemma 11.** *Let $u_n \in \mathbb{C}$ for $n = 1, \ldots, N$. Then*

$$\left|\prod_{n=1}^{N}(1+u_n) - 1\right| \le \exp\left(\sum_{n=1}^{N}|u_n|\right) - 1.$$

Let us begin with the Main Distortion Lemma concerning control of the space derivative in a neighbourhood of the hyperbolic set.

**Lemma 12.** *For every $\varepsilon > 0$, we can find $\delta' > 0$ and $r > 0$ arbitrarily small with the following property: for any $a, b \in B(\lambda_0, r)$, if $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for all $k \le n$ and $\lambda = a, b$, then*

$$\left|\frac{(f_a^n)'(e_a)}{(f_b^n)'(e_b)} - 1\right| < \varepsilon.$$

*Proof.* First, we will show that for an arbitrarily small $\varepsilon_1$ it is possible to choose $\delta' > 0$ so that

$$\left|\frac{(f_\lambda^n)'(\mu_0(\lambda))}{(f_\lambda^n)'(\xi_0(\lambda))} - 1\right| \le \varepsilon_1 \tag{4}$$

provided $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for all $k \le n$.

By the expansion property, and since $|f_\lambda'| > C_\delta^{-1}$ on $\mathcal{N}$ for some $C_\delta > 0$, we can estimate for any $\lambda \in B(\lambda_0, r)$

$$\begin{aligned}\sum_{j=0}^{n-1}\left|\frac{f_\lambda'(\mu_j(\lambda)) - f_\lambda'(\xi_j(\lambda))}{f_\lambda'(\xi_j(\lambda))}\right| &\le C_\delta \sum_{j=0}^{n-1}\left|f_\lambda'(\mu_j(\lambda)) - f_\lambda'(\xi_j(\lambda))\right| \\ &\le C_\delta \max_{z\in\mathcal{N}}|f_\lambda''(z)| \sum_{j=0}^{n-1}|\mu_j(\lambda) - \xi_j(\lambda)| \\ &\le \tilde{C}\sum_{j=0}^{n-1} Ca^{j-n}|\mu_n(\lambda) - \xi_n(\lambda)| \le C'\delta',\end{aligned}$$

where $\max|f_\lambda''(z)|$ is bounded on $B(\lambda_0, r)$, since $\mathcal{N}$ contains no poles of $f_\lambda^j$ for $j = 1, \ldots, N$ and $\lambda \in B(\lambda_0, r)$. Using Lemma 11, we obtain the inequality (4) if $\delta' > 0$ is small enough.

Secondly, for any $\varepsilon_2 > 0$, if $\delta' > 0$ and $r > 0$ are chosen sufficiently small, then for every $t, s \in B(\lambda_0, r)$,

$$\left|\frac{(f_t^n)'(\mu_0(t))}{(f_s^n)'(\mu_0(s))} - 1\right| \le \varepsilon_2. \tag{5}$$

Put $a_{\lambda,j} = f_\lambda'(\mu_j(\lambda))$. Since each $a_{\lambda,j}$ is analytic with respect to $\lambda$, it can be expressed as follows: $a_{\lambda,j} = a_{\lambda_0,j}(1 + c_j(\lambda - \lambda_0)^l + \ldots)$. Moreover, by Lemma 10 and (2) we have

$$n \le -C\log|x(\lambda)| \le -\tilde{C}\log|\lambda - \lambda_0|, \tag{6}$$

where constants depend only on $\delta'$ and not on $n$. Thus, if $c = \sum_{j=0}^{n-1} c_j$,

$$\frac{(f_t^n)'(\mu_0(t))}{(f_s^n)'(\mu_0(s))} = \prod_{j=0}^{n-1}\frac{a_{t,j}}{a_{s,j}} = \prod_{j=0}^{n-1}\frac{a_{\lambda_0,j}(1 + c_j(t-\lambda_0)^l + \ldots)}{a_{\lambda_0,j}(1 + c_j(s-\lambda_0)^l + \ldots)} = \frac{1 + cn(t-\lambda_0)^l + \ldots}{1 + cn(s-\lambda_0)^l + \ldots}.$$

Now, both the numerator and the denominator can be made arbitrarily close to one if only $r > 0$ is small enough, since they are of orders $1 + \mathcal{O}(|t-\lambda_0|^l \log|t-\lambda_0|)$ and $1 + \mathcal{O}(|s-\lambda_0|^l \log|s-\lambda_0|)$, respectively.

Putting together (4) and (5) we complete the proof. □

Next, we compare space and parameter derivatives.

**Lemma 13.** *Let $\varepsilon > 0$. If $\delta' > 0$ is sufficiently small, then for every $0 < \delta'' < \delta'$, there exists $r > 0$ such that the following holds: for any $\lambda \in B(\lambda_0, r)$, if $|\xi_k(\lambda) - \mu_k(\lambda)| \le \delta'$ for $k \le n$ and $|\xi_n(\lambda) - \mu_n(\lambda)| \ge \delta''$, then*

$$\left|\frac{\xi_n'(\lambda)}{(f_\lambda^n)'(\mu_0(\lambda))x'(\lambda)} - 1\right| \le \varepsilon.$$

*Proof.* Note that

$$\xi_n(\lambda) = \mu_n(\lambda) + (f_\lambda^n)'(\mu_0(\lambda))x(\lambda) + E_n(\lambda), \tag{7}$$

where $|E_n(\lambda)| \le \varepsilon_1 |\xi_n(\lambda) - \mu_n(\lambda)|$ independently of $n$, for any small $\varepsilon_1 > 0$, if only $\delta' > 0$ was chosen small enough. To verity this we will proceed similarly as in the first part of the proof of Lemma 12. First, we can write

$$\frac{(f_\lambda^n)'(\mu_0(\lambda))x(\lambda)}{\xi_n(\lambda) - \mu_n(\lambda)} = \prod_{j=0}^{n-1} \frac{f_\lambda'(\mu_j(\lambda))(\xi_j(\lambda) - \mu_j(\lambda))}{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)}.$$

By the expansion property (Lemma 10) we can estimate as follows:

$$\begin{aligned}
\left| \frac{f_\lambda'(\mu_j(\lambda))(\xi_j(\lambda) - \mu_j(\lambda))}{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)} - 1 \right| &\le \frac{1}{Ca} \left| f_\lambda'(\mu_j(\lambda)) - \frac{\xi_{j+1}(\lambda) - \mu_{j+1}(\lambda)}{\xi_j(\lambda) - \mu_j(\lambda)} \right| \\
&\le \frac{1}{Ca} \max_{z \in \mathcal{N}} |f_\lambda''(z)| \, |\xi_j(\lambda) - \mu_j(\lambda)| \\
&\le \frac{M''}{Ca} C^{-1} a^{j-n} |\xi_n(\lambda) - \mu_n(\lambda)|,
\end{aligned}$$

for $M'' = \max\{|f_\lambda''(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, r)\}$, which is finite by the choice of $\mathcal{N}$. Applying Lemma 11, we obtain the desired estimate.

Put again $f_\lambda'(\mu_j(\lambda)) = a_{\lambda,j}$, then $(f_\lambda^n)'(\mu_0(\lambda)) = \prod_{j=0}^{n-1} a_{\lambda,j}$. Now, differentiate $\xi_n$ with respect to $\lambda$. By the Chain Rule, we get

$$\begin{aligned}
\xi_n'(\lambda) =& \mu_n'(\lambda) + x'(\lambda) \prod_{j=0}^{n-1} a_{\lambda,j} + x(\lambda) \sum_{j=0}^{n-1} a_{\lambda,j}' \frac{\prod_{k=0}^{n-1} a_{\lambda,k}}{a_{\lambda,j}} + E_n'(\lambda) \\
=& \prod_{j=0}^{n-1} a_{\lambda,j} \left( x'(\lambda) + x(\lambda) \sum_{j=0}^{n-1} \frac{a_{\lambda,j}'}{a_{\lambda,j}} + \frac{\mu_n'(\lambda) + E_n'(\lambda)}{\prod_{j=0}^{n-1} a_{\lambda,j}} \right).
\end{aligned}$$

In the following, we want to show that $x'(\lambda)$ is the leading term in the above expression.

Recall that $\delta'' \le |\xi_n(\lambda) - \mu_n(\lambda)| \le \delta'$. Thus, by (7) and the estimate on $|E_n(\lambda)|$ we have

$$(1 - \varepsilon_1)\delta'' \le |x(\lambda)| \prod_{j=0}^{n-1} |a_{\lambda,j}| \le (1 + \varepsilon_1)\delta'. \tag{8}$$

Now, we need to estimate $|\sum \frac{a_{\lambda,j}'}{a_{\lambda,j}}|$. Note that, since $\mu_j(\lambda) = f_\lambda^j(\mu_0(\lambda)) \in \mathcal{H}_\lambda$, we get

$$|a_{\lambda,j}| = |f_\lambda'(\mu_j(\lambda))| \le \max_{z \in \mathcal{H}_\lambda, \lambda \in B(\lambda_0, r)} |f_\lambda'(z)| \quad \text{and} \quad |a_{\lambda,j}| \ge Ca, \; C, a > 0.$$

Since $a_{\lambda,j}$ are uniformly bounded for every $j$ and $\lambda \in B(\lambda_0, r)$, then, by Cauchy's formula, also $a_{\lambda,j}'$ are uniformly bounded by some $M' > 0$ on a slightly smaller ball $B(\lambda_0, r')$. We get

$$\left| \sum_{j=0}^{n-1} \frac{a_{\lambda,j}'}{a_{\lambda,j}} \right| \le \sum_{j=0}^{n-1} \left| \frac{a_{\lambda,j}'}{a_{\lambda,j}} \right| \le n \frac{M'}{Ca} =: n\tilde{C}.$$

Thus, using (6), we get

$$|x(\lambda)|\left|\sum \frac{a'_{\lambda,j}}{a_{\lambda,j}}\right| \leq |x(\lambda)|n\tilde{C} \leq |x(\lambda)|C'(-\log|x(\lambda)|)\tilde{C},$$

where $C' > 0$ depends only on $\delta'$. Moreover, up to a multiplicative constant,

$$\frac{-|x(\lambda)|\log|x(\lambda)|}{|x'(\lambda)|} \asymp \frac{-|(\lambda-\lambda_0)^K|\log|\lambda-\lambda_0|}{|(\lambda-\lambda_0)^{K-1}|} \asymp -|\lambda-\lambda_0|\log|\lambda-\lambda_0|. \tag{9}$$

Let us estimate

$$\begin{aligned}\frac{\xi'_n(\lambda)}{(f^n_\lambda)'(\mu_0(\lambda))x'(\lambda)} - 1 =& \frac{\prod a_{\lambda,j}\left(x'(\lambda)+x(\lambda)\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}}+\frac{\mu'_n(\lambda)+E'_n(\lambda)}{\prod a_{\lambda,j}}\right)}{\prod a_{\lambda,j}x'(\lambda)} - 1\\ =& \frac{x(\lambda)\sum\frac{a'_{\lambda,j}}{a_{\lambda,j}}}{x'(\lambda)} + \frac{\mu'_n(\lambda)+E'_n(\lambda)}{\prod a_{\lambda,j}x'(\lambda)}.\end{aligned}$$

By (9), the first summand tends uniformly to zero as $\lambda \to \lambda_0$. To see what happens with the second summand, note that $|\mu'_n(\lambda)+E'_n(\lambda)|$ is uniformly bounded by Cauchy's formula, since $\mu_n(\lambda)$ and $E_n(\lambda)$ are bounded. We have also seen that $|\prod a_{\lambda,j}x(\lambda)|$ is bounded (from both sides) independently of $n$. Therefore, by (8), we get

$$\left|\frac{1}{\prod a_{\lambda,j}x'(\lambda)}\right| = \left|\frac{1}{\prod a_{\lambda,j}x(\lambda)}\right|\left|\frac{x(\lambda)}{x'(\lambda)}\right| \leq \frac{1}{\delta''(1-\varepsilon_1)}\left|\frac{x(\lambda)}{x'(\lambda)}\right| \asymp |\lambda-\lambda_0|,$$

thus also the second summand tends uniformly to zero as $\lambda \to \lambda_0$. This completes the proof. □

Combining Lemma 12 and Lemma 13, we obtain the following result.

**Corollary 14.** *Let $\varepsilon > 0$. If $\delta' > 0$ is small enough and $0 < \delta'' < \delta'$, we can find $r > 0$ such that for every $\lambda \in B(\lambda_0, r)$, if $|\xi_k(\lambda)-\mu_k(\lambda)| \leq \delta'$ for $k \leq n$ and $|\xi_n(\lambda)-\mu_n(\lambda)| \geq \delta''$, then*

$$\left|\frac{\xi'_n(\lambda)}{(f^n_\lambda)'(e_\lambda)x'(\lambda)} - 1\right| \leq \varepsilon.$$

## 3.3. DISTORTION IN AN ANNULUS

As we have seen in the previous section, we need to move away from $\lambda_0$ in the parameter ball $B(\lambda_0, r)$ in order to have nice distortion estimates. That is why we will restrict our considerations to an annular domain. This approach gives us a powerful tool, which is bounded distortion of $\xi_n$, and leads to the control of the growth of $B(\lambda_0, r)$ under $\xi_n$.

Consider an annulus in the parameter space

$$A = A(\lambda_0; r_1, r_2) = \{\lambda : r_1 < |\lambda - \lambda_0| < r_2\}.$$

Note that, by (2), for some constant $C \geq 1$ and any $\lambda_1, \lambda_2 \in A$,

$$C^{-1}\left(\frac{r_1}{r_2}\right)^{K-1} \leq \left|\frac{x'(\lambda_1)}{x'(\lambda_2)}\right| \leq C\left(\frac{r_2}{r_1}\right)^{K-1},$$

where $K$ is the degree of $x(.)$ at $\lambda_0$. Therefore, from Corollary 14 and Lemma 12, we conclude that if $r_2 > 0$ is small enough, then

$$\tilde{C}^{-1}\left(\frac{r_1}{r_2}\right)^{K-1} \leq \left|\frac{\xi_n'(\lambda_1)}{\xi_n'(\lambda_2)}\right| \leq \tilde{C}\left(\frac{r_2}{r_1}\right)^{K-1}$$

for some $\tilde{C} \geq 1$ and all $\lambda_1, \lambda_2 \in A$, as long as $|\xi_k(\lambda) - \mu_k(\lambda)| \leq \delta'$ for $k \leq n$ and $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$ for all $\lambda \in A$.

**Lemma 15.** *Let $\varepsilon > 0$. If $\delta' > 0$ and $\frac{\delta''}{\delta'}$ are sufficiently small and $0 < \delta'' < \delta'$, then there exists an $r > 0$ such that any parameter ball $B = B(\lambda_0, r_2) \subset B(\lambda_0, r)$ has the following property: Let $n$ be maximal such that $|\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta'$ for all $\lambda \in B$. Let $r_1 < r_2$ be minimal such that $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$ for all $\lambda \in A = A(\lambda_0; r_1, r_2)$. Then $\frac{r_1}{r_2} \leq \frac{1}{10}$ and there exists some $\delta_1'$, $\delta'' < \delta_1' < \delta'$, such that*

$$A\left(\mu_n(\lambda_0); \delta''(1+\varepsilon), \delta_1'(1-\varepsilon)\right) \subset \xi_n(A) \subset A\left(\mu_n(\lambda_0); \delta''(1-\varepsilon), \delta_1'(1+\varepsilon)\right).$$

*Moreover, $\xi_n$ is at most $K$-to-1 on $B$.*

*Proof.* Note that the parameter circle $\gamma_r = \{\lambda : |\lambda - \lambda_0| = r\}$ for small $r > 0$ is mapped under $x(.)$ onto a curve that encircles $\lambda_0$ $K$ times so that $x(\gamma_r)$ is close to a circle of radius $\alpha_K r^K$. Moreover, $|\mu_n(\lambda) - \mu_n(\lambda_0)| = |h_\lambda(f_{\lambda_0}^n(e_{\lambda_0})) - f_{\lambda_0}^n(e_{\lambda_0})|$ is arbitrarily small for small radii in the parameter space, since $\mathcal{H}$ and $\mathcal{H}_\lambda$ can be as close to each other as desired for $\lambda \in B(\lambda_0, r)$. Thus, if $r$ is small and $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$, then

$$|\xi_n(\lambda) - \mu_n(\lambda)| > P|\mu_n(\lambda) - \mu_n(\lambda_0)| \tag{10}$$

for some big $P \gg 1$ depending only on $\delta''$ and $r$. Arguing again like in the proof of Lemma 13, we get for every $\varepsilon_1 > 0$ we can choose $\delta' > 0$ and $r > 0$ so that

$$|\xi_n(\lambda) - \mu_n(\lambda) - (f_\lambda^n)'(e_\lambda)x(\lambda)| < \varepsilon_1|\xi_n(\lambda) - \mu_n(\lambda)| \tag{11}$$

for all $\lambda \in B(\lambda_0, r)$.

If $r_1$ is minimal such that $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$ for all $\lambda \in A(\lambda_0; r_1, r_2)$, then for some $\lambda_1$ with $|\lambda_1 - \lambda_0| = r_1$ we have

$$|\xi_n(\lambda_1) - \mu_n(\lambda_1)| = \delta''. \tag{12}$$

On the other hand, from the definition of $n$, we obtain that for some $\lambda_2$ with $|\lambda_2 - \lambda_0| = r_2$, $|\xi_{n+1}(\lambda_2) - \mu_{n+1}(\lambda_2)| \geq \delta'$. But

$$|\xi_{n+1}(\lambda_2) - \mu_{n+1}(\lambda_2)| = |f_{\lambda_2}(\xi_n(\lambda_2)) - f_{\lambda_2}(\mu_n(\lambda_2))| \leq M'|\xi_n(\lambda_2) - \mu_n(\lambda_2)|,$$

where $M' = \max\{|f'_\lambda(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, r)\}$, which is finite since $\mathcal{N}$ contains neither poles nor essential singularities of $f_\lambda$. Therefore, we get

$$|\xi_n(\lambda_2) - \mu_n(\lambda_2)| \geq \frac{\delta'}{M'}. \tag{13}$$

Moreover, by (11), for every $\lambda \in B(\lambda_0, r)$, if $r > 0$ and $\delta' > 0$ are small enough, then

$$\frac{1}{1+\varepsilon_1}|(f^n_\lambda)'(e_\lambda)x(\lambda)| \leq |\xi_n(\lambda) - \mu_n(\lambda)| \leq \frac{1}{1-\varepsilon_1}|(f^n_\lambda)'(e_\lambda)x(\lambda)|. \tag{14}$$

Using (12), (13), (14) and Lemma 12 we can estimate as follows:

$$\frac{\delta'}{\delta''} \leq \frac{M'|\xi_n(\lambda_2) - \mu_n(\lambda_2)|}{|\xi_n(\lambda_1) - \mu_n(\lambda_1)|} \leq M'\frac{1+\varepsilon_1}{1-\varepsilon_1}\left|\frac{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_2)}{(f^n_{\lambda_1})'(e_{\lambda_1})x(\lambda_1)}\right| \leq M'\frac{(1+\varepsilon_1)^2}{1-\varepsilon_1}\left|\frac{x(\lambda_2)}{x(\lambda_1)}\right|.$$

Thus, we can choose $\delta'' > 0$ so small that $\frac{r_1}{r_2} \leq \frac{1}{10}$ independently of $n$.

Now, we want to see how many times $\xi_n(\lambda) - \mu_n(\lambda)$ orbits around 0 as the parameter $\lambda$ moves along the circle $\gamma_r$, $r > r_1$. To establish this, let us look at the expression $\frac{\xi_n(\lambda)-\mu_n(\lambda)}{|\xi_n(\lambda)-\mu_n(\lambda)|}$. By (11) we have

$$\left|\frac{\xi_n(\lambda) - \mu_n(\lambda)}{|\xi_n(\lambda) - \mu_n(\lambda)|} - \frac{(f^n_\lambda)'(e_\lambda)x(\lambda)}{|\xi_n(\lambda) - \mu_n(\lambda)|}\right| \leq \varepsilon_1,$$

so it is the same to ask how many times $(f^n_\lambda)'(e_\lambda)x(\lambda)$ encircles 0. By Lemma 12, $(f^n_\lambda)'(e_\lambda)$ is essentially constant on $B(\lambda_0, r_2)$, so the number we are looking for is $K$, the same as for $x(\lambda)$. Furthermore, (10) implies that $|\mu_n(\lambda) - \mu_n(\lambda_0)|$ is much smaller than $|\xi_n(\lambda) - \mu_n(\lambda)|$. This means that $\xi_n(\lambda)$ orbits around $\mu_n(\lambda_0) = \xi_n(\lambda_0)$ also $K$ times, close to some circle centered at $\mu_n(\lambda_0)$. By the Argument Principle, the degree of $\xi_n$ is at most $K$.

In order to prove that the shape of the considered set is really close to round, let us take $\lambda_1, \lambda_2$ with $|\lambda_1 - \lambda_0| = |\lambda_2 - \lambda_0| = r$. Then, again by (14) and Lemma 12, we obtain the following estimates

$$\begin{aligned}\left|\frac{\xi_n(\lambda_1) - \mu_n(\lambda_0)}{\xi_n(\lambda_2) - \mu_n(\lambda_0)}\right| &\leq \frac{1+\varepsilon}{1-\varepsilon}\left|\frac{\xi_n(\lambda_1) - \mu_n(\lambda_1)}{\xi_n(\lambda_2) - \mu_n(\lambda_2)}\right| \leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\left|\frac{(f^n_{\lambda_1})'(e_{\lambda_1})x(\lambda_1)}{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_2)}\right| \\ &\leq \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2}\left|\frac{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_1)}{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_2)}\right| = \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2}\left|\frac{x(\lambda_1)}{x(\lambda_2)}\right|.\end{aligned}$$

For $r$ small enough, last expression can be arbitrarily close to 1 independently of $n$. This means that the set $\xi_n(\gamma_r)$ is close to a circle centered at $\xi_n(\lambda_0) = \mu_n(\lambda_0)$ and of radius $|\xi_n(\lambda) - \mu_n(\lambda_0)|$ for any $|\lambda - \lambda_0| = r$, so the annulus $A$ is mapped onto a slightly distorted annulus, whose shape can be controlled independently of $n$. This completes the proof of the lemma. □

Using the notation of the previous lemma, we obtain from its proof and Lemma 12 the following important corollary:

**Corollary 16.** *If $\delta' > 0$ and $r > 0$ are small enough and $n$ is maximal such that $|\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta'$ for all $\lambda \in B(\lambda_0, r_2)$, then for all $\lambda' \in B(\lambda_0, r_2)$ satisfying $|\lambda' - \lambda_0| = r_2$ we have $|\xi_n(\lambda') - \mu_n(\lambda')| \geq \frac{\delta'}{2M'}$.*

## 3.4. MEASURE ESTIMATES

By now we know how to control the behaviour of $\xi_n$ in a small scale. In this section, we will derive measure estimates in a large scale, i.e. when a parametric ball attains some fixed size under $\xi_n$. Recall that we consider $f_\lambda$, $\lambda \in B(\lambda_0, \varepsilon)$, for some small $\varepsilon > 0$ and that $\lambda_0$ is the parameter satisfying assumptions of Theorem 3. Assuming that $r \leq \varepsilon$ is so small that $z$ and its holomorphic motion $h_\lambda(z)$ are close enough for all $z \in \mathcal{H}$ and $\lambda \in B(\lambda_0, r)$, from Lemma 15 and Corollary 16 we get the following fact:

**Proposition 17.** *There exist $\delta' > 0$ and $0 < r < \varepsilon$, depending only on $f_{\lambda_0}$, such that for any $0 < r_2 < r$, if $n$ is the biggest number for which $\operatorname{diam}(\xi_n(B(\lambda_0, r_2))) \leq \delta'$, then we can find two discs $D_1$ i $D_2$ such that $D_1 \subset D \subset D_2$, where $D = \xi_n(B(\lambda_0, r_2))$, with the following properties*

$$\frac{\operatorname{diam}(D_2)}{\operatorname{diam}(D_1)} = 4M', \quad \operatorname{diam}(D_1) = \frac{\delta'}{M'}$$

*and $D_1$ is centered at $\mu_n(\lambda_0) \in J(f_{\lambda_0})$. The degree of $\xi_n$ on $B(\lambda_0, r)$ is bounded above by $K$, depending only on the family $f_\lambda$, $\lambda \in B(\lambda_0, \varepsilon)$.*

The next step is to estimate the Lebesgue measure of the set of those parameters $\lambda$ for which some iterate $f_\lambda^n(e_\lambda)$ either turns back to a neighbourhood of a critical point or escapes close to infinity. First, however, we need to know how many iterates are required to cover a neighbourhood of infinity and critical points

$$U_\delta = B\big(\operatorname{Crit}(f_{\lambda_0}), \delta\big) \cup B(\infty, \delta), \tag{15}$$

for an arbitrarily small $\delta > 0$. To be more precise, we want to estimate the number of iterates of $f_\lambda$, $\lambda \in B(\lambda_0, r)$ for some $r > 0$, after which the image of a small disk intersecting the Julia set covers $U_\delta$.

Recall that the Julia set $J(f_\lambda)$ is the closure of the prepoles of $f_\lambda$ (see e.g. [3]), thus any open disc intersecting the Julia set after a finite number of steps will cover under $f_\lambda$ the whole $\overline{\mathbb{C}}$ (elliptic functions have no omitted values). Moreover, since poles move holomorphically with the parameter $\lambda$, the number of steps is locally constant in the parameter plane.

**Lemma 18.** *Let $D$ be an open and bounded set disjoint from $U_\delta$, containing an open disk of radius $d > 0$ centered at the Julia set of some $f = f_\lambda$. Then we can choose an $N$, depending only on $d$, $f$ and $U_\delta$, such that*

$$\inf\left\{m \in \mathbb{N} : f^m(D) \supset \overline{U_\delta}\right\} \leq N.$$

*Proof.* Cover $\overline{J(f) \setminus U_\delta}$ with a collection of open disks $D_z$ of diameter $d$ centered at $z \in \overline{J(f) \setminus U_\delta}$. Since the prepoles of $f$ are dense in $J(f)$, for every $D_z$ there is a minimal $n = n(z)$ such that

$$f^n(D_z) \supset \overline{U_\delta}.$$

Since $f^n$ is continuous, $n(z)$ is constant in some neighbourhood of $z$. Moreover, $\overline{J(f) \setminus U_\delta}$ is compact in $\mathbb{C}$, hence we can find an integer $N$ such that $n(z) \leq N$ for every $z$. □

Note that we can choose $r > 0$ so that the statement holds for every $f_\lambda$, $\lambda \in B(\lambda_0, r)$ and possibly slightly bigger $N$, which depends only on $d > 0$ for $r$ small enough. It is possible since the dependence on $\lambda$ is analytic, hence continuous.

We know now that $f^m(D) \Supset U_\delta$ for some $m \leq N$. We will estimate the measure of those points from $D$ that get mapped into $U_\delta$ under $f^j$ for some $j \leq m$. Recall that $f = f_\lambda$ is a Weierstrass elliptic function and $D$ is an open and bounded set disjoint from $U_\delta$. In particular, $D \cap B(\infty, \delta) = \emptyset$. The following lemma is similar to an analogous one in the exponential case [2], however, because of the presence of poles, we need to be much more careful. Let $\mu$ denotes the Lebesgue measure on the Riemann sphere $\overline{\mathbb{C}}$ and recall that the derivatives are spherical and $U_\delta$ is given by (15).

**Lemma 19.** *Assume that $D$ is an open set disjoint from $U_\delta$ and $f^m(D) \Supset U_\delta$ for some integer $m$. Then there exists a constant $C > 0$, depending only on $f$, $m$ and $U_\delta$, such that*

$$\mu\left(\left\{z \in D\colon\ f^j(z) \in U_\delta \textit{ for some } 1 \leq j \leq m\right\}\right) \geq C\mu(D).$$

*Proof.* Let us define

$$F = \{z \in D : f^j(z) \in U_\delta \text{ for some } 1 \leq j \leq m\}$$

Divide $F$ into $m$ pairwise disjoint subsets, i.e., the following domains of the first entry map to $U_\delta$:

$$\begin{aligned}
F_1 &= \{z \in D : f(z) \in U_\delta\} = f^{-1}(U_\delta) \cap D,\\
F_2 &= \{z \in D : f^2(z) \in U_\delta \text{ but } f(z) \notin U_\delta\} = f^{-2}(U_\delta) \cap f^{-1}\left(\overline{\mathbb{C}} \setminus U_\delta\right) \cap D,\\
F_3 &= \{z \in D : f^3(z) \in U_\delta \text{ but } f(z) \notin U_\delta, f^2(z) \notin U_\delta\},\\
&\ \vdots\\
F_m &= \{z \in D : f^m(z) \in U_\delta \text{ but } f^j(z) \notin U_\delta \text{ for } j \leq m-1\}\\
&= f^{-m}(U_\delta) \cap \bigcap_{j=1}^{m-1} f^{-j}\left(\overline{\mathbb{C}} \setminus U_\delta\right) \cap D.
\end{aligned}$$

Then, obviously, $F = F_1 \cup F_2 \cup \ldots \cup F_m$ and the sets $F_1, \ldots, F_m$ are pairwise disjoint. Moreover, since $D$ is bounded, the definition assures that no $F_j$ contains an essential singularity of $f^j$, so the spherical derivative of $f^j$ is well defined everywhere in $F_j$ for $j = 1, \ldots, m$. Notice also that

$$D \setminus F = \{z \in D : f(z) \notin U_\delta, \ldots, f^m(z) \notin U_\delta\} = \bigcap_{j=1}^{m} f^{-j}\left(\overline{\mathbb{C}} \setminus U_\delta\right) \cap D.$$

Since $\overline{\mathbb{C}} \setminus U_\delta$ is bounded, the set $D \setminus F$ contains no poles of any $f^j$ for $j = 1, \ldots, m$, hence, also, no essential singularity of $f^m$.

To estimate the degree of $f^m$ on $D \setminus F$, recall that $f$ is periodic with respect to an appropriate lattice and on every period parallelogram the degree of $f$ equals two. The set $\overline{\mathbb{C}} \setminus U_\delta$ is bounded in $\mathbb{C}$, i.e. it is contained in $\overline{\mathbb{C}} \setminus B(\infty, \delta)$, so it intersects finitely many, say $n_\delta$, period parallelograms. Hence, the degree of $f$ on $\overline{\mathbb{C}} \setminus U_\delta$ is bounded by $2n_\delta$. Now, every iterate of $f$ that we consider, maps a subset of $\overline{\mathbb{C}} \setminus U_\delta$ back into $\overline{\mathbb{C}} \setminus U_\delta$. Thus, the degree of $f^2$ is bounded by $(2n_\delta)^2$ on the set $f^{-1}\left(\overline{\mathbb{C}} \setminus U_\delta\right) \cap (\overline{\mathbb{C}} \setminus U_\delta)$, etc. We conclude that the degree of $f^m$ on $D \setminus F$ is at most $(2n_\delta)^m$ and this number depends only on $f$, $m$ and $\delta$.

Moreover, on every $F_j$, the spherical derivative $|(f^j)'|$ is bounded from above by some constant $c_j = c_j(f, m, \delta)$. On the other hand. on $D \setminus F$ the quantity $|(f^m)'|$ is bounded from below by a constant $a = a(f, m, \delta) > 0$ (there are neither poles nor essential singularities of $f^m$ and we are far away from $\mathrm{Crit}(f^m)$). We get the following estimates:

$$\mu(U_\delta) \le \sum_{j=1}^{m} \int_{F_j} |(f^j)'(z)|^2 d\mu(z) \le \sum_{j=1}^{m} c_j^2\, \mu(F_j) \le \max_{j=1,\ldots,m} c_j^2 \sum_{j=1}^{m} \mu(F_j) =: C_1 \mu(F). \tag{16}$$

Denote $g(w) = \{z \in D \setminus F : f^m(z) = w\}$ for $w \in \overline{\mathbb{C}} \setminus U_\delta$. Then

$$\mu(D \setminus F) = \int_{\overline{\mathbb{C}} \setminus U_\delta} \sum_{z \in g(w)} |(f^m)'(z)|^{-2} d\mu(w) \le (2n_\delta)^m a^{-2} \mu\left(\overline{\mathbb{C}} \setminus U_\delta\right) =: \kappa\, \mu\left(\overline{\mathbb{C}} \setminus U_\delta\right). \tag{17}$$

Finally, for some constant $M_\delta$ depending only on $\delta$, we have

$$\mu(U_\delta) \ge M_\delta\, \mu\left(\overline{\mathbb{C}} \setminus U_\delta\right). \tag{18}$$

Combining (16), (17) and (18) we obtain

$$\mu(F) \ge \frac{1}{C_1}\, \mu(U_\delta) \ge \frac{M_\delta}{C_1} \mu(\overline{\mathbb{C}} \setminus U_\delta) \ge \frac{M_\delta}{C_1 \kappa}\, \mu(D \setminus F),$$

which implies that

$$\mu(F) \ge C\mu(D)$$

for some constant $C = C(f, m, \delta)$. □

## 3.5. CONCLUSION

To conclude the proof of Theorem 7, recall that $f_{\lambda_0}$ is a Weierstrass elliptic function from $\mathcal{W}_t$ or $\mathcal{W}_s$ with $\lambda_0 \in \mathcal{M}$ and consider nearby maps $f_\lambda$, $\lambda \in B(\lambda_0, r)$, for some small $r > 0$. Take an arbitrarily small $\delta > 0$ (e.g. such that $\lambda_0 \in \mathcal{M}_\delta$). We want to show that the set $\mathcal{M}_\delta$ has the Lebesgue density less than one at $\lambda_0$.

Assume that $r > 0$ is so small that critical points of $f_\lambda$, $\lambda \in B(\lambda_0, r)$, are $\delta/4$ close to appropriate critical points of $f_{\lambda_0}$ – it is possible since critical points depend analytically on $\lambda$ and we have only finitely many periodic families of critical points for Weierstrass elliptic functions. Then we have

$$\forall_{\lambda \in B(\lambda_0, r)} \qquad U_{3\delta/4} \subset B\big(\mathrm{Crit}(f_\lambda), \delta\big) \cup B(\infty, \delta), \tag{19}$$

where $U_\delta$ is given by (15). In what follows, we will estimate the Lebesgue measure of the set of parameters $\lambda$, for which some iterate of a critical value $e_\lambda$ falls into $U_{3\delta/4}$, hence $\lambda \notin \mathcal{M}_\delta$.

Let $\delta' > 0$ and $r > 0$ be chosen so that the statement of Proposition 17 is satisfied and all our expansion and distortion properties hold. Consider a parameter ball $B = B(\lambda_0, r_2)$ for any $r_2 \leq r$ and let $n$ be the largest integer for which the set $D := \xi_n(B)$ has the diameter at most $\delta'$. Let the discs $D_1 \subset D \subset D_2$ be as in Proposition 17.

Lemma 18 implies that there exists an $N > 0$ such that $f_{\lambda_0}^m(D_1) \Supset U_{\delta/2}$ for some $m \leq N$, independently of the center of $D_1$. Because of the inclusions $D_1 \subset D \subset D_2$ and since $\mathrm{diam}(D_2)/\mathrm{diam}(D_1) = 4M'$ we get, by Lemma 19,

$$\mu\left(\left\{z \in D : f_{\lambda_0}^m(z) \in U_{\delta/2}\right\}\right) \geq C_1 \mu(D) \tag{20}$$

for some constant $C_1$ depending only on the family $f_\lambda$, the set $U_\delta$ and $N$. Since we have only finitely many steps to consider, we can decrease, if necessary, the radius $r > 0$ so that for every $\lambda \in B(\lambda_0, r)$,

$$f_{\lambda_0}^m(\xi_n(\lambda)) \in U_{\delta/2} \quad \Longrightarrow \quad \xi_{n+m}(\lambda) = f_\lambda^m(\xi_n(\lambda)) \in U_{3\delta/4}$$

for any $m \leq N$.

**Lemma 20.** *It is possible to choose $\delta'' \in (0, \delta')$ so that for every $r_2$, $0 < r_2 < r$ and for every $\lambda \in B(\lambda_0, r_2)$*

$$\xi_{n+j}(\lambda) \in U_{3\delta/4} \text{ for some } j \leq N \quad \Longrightarrow \quad \lambda \in A(\lambda_0; r_1, r_2),$$

*where $r_1 > 0$ is minimal for which $|\xi_n(\lambda) - \mu_n(\lambda)| \geq \delta''$ for all $\lambda \in A(\lambda_0; r_1, r_2)$.*

*Proof.* We can choose $\delta'' > 0$ as small as desired, provided $r > 0$ is small enough. Thus, to ensure that for any $\lambda \in B(\lambda_0, r)$ with $|\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta''$ and for all $j \leq N$

$$|\xi_{n+j}(\lambda) - \mu_{n+j}(\lambda)| \leq b^j |\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta'$$

it is sufficient to choose $\delta''$ so small that $b^N \leq \frac{\delta'}{\delta''}$, where

$$b = \max\{|f'_\lambda(z)| : z \in \mathcal{N}, \lambda \in B(\lambda_0, r)\}, \; 1 < b < \infty.$$

Next, we know that $\mu_{n+j}(\lambda) \in \mathcal{H}_\lambda \subset \mathcal{N}$ (if $r$ is small) and $\mathcal{N} \cap U_\delta = \emptyset$. Therefore, if $\delta' < \delta/4$, then $\xi_{n+j}(\lambda) \notin U_{3\delta/4}$ for all $\lambda$ satisfying $|\xi_n(\lambda) - \mu_n(\lambda)| \leq \delta''$. $\square$

We get the following inclusions:

$$A(\lambda_0; r_1, r_2) \supset \left\{\lambda \in B : \xi_{n+m}(\lambda) \in U_{3\delta/4}\right\} \supset \xi_n^{-1}\left(\left\{z \in D : f^m_{\lambda_0}(z) \in U_{\delta/2}\right\}\right). \tag{21}$$

Recall that inside the annulus $A = A(\lambda_0; r_1, r_2)$ we have the bounded distortion of $\xi_n$, i.e.

$$\frac{1}{C'}\left(\frac{r_1}{r_2}\right)^{K-1} \leq \left|\frac{\xi'_n(\lambda_1)}{\xi'_n(\lambda_2)}\right| \leq C'\left(\frac{r_2}{r_1}\right)^{K-1}.$$

Moreover, if $r > 0$ is small enough and $|\lambda_i - \lambda_0| = r_i$, $i = 1, 2$, then since $\operatorname{diam}(\xi_n(B)) \leq \delta'$,

$$|\xi_n(\lambda_2) - \mu_n(\lambda_2)| \leq \frac{1}{1-\varepsilon}\delta',$$

and, by the choice of $r_1$,

$$|\xi_n(\lambda_1) - \mu_n(\lambda_1)| \geq \delta''.$$

Consequently, applying Lemma 12 and (11), we get, like in the proof of Lemma 15

$$\begin{aligned}
\frac{\delta''}{\delta'} &\leq \frac{1}{1-\varepsilon}\left|\frac{\xi_n(\lambda_1) - \mu_n(\lambda_1)}{\xi_n(\lambda_2) - \mu_n(\lambda_2)}\right| \leq \frac{1+\varepsilon}{(1-\varepsilon)^2}\left|\frac{(f^n_{\lambda_1})'(e_{\lambda_1})x(\lambda_1)}{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_2)}\right| \\
&\leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\left|\frac{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_1)}{(f^n_{\lambda_2})'(e_{\lambda_2})x(\lambda_2)}\right| = \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\left|\frac{x(\lambda_1)}{x(\lambda_2)}\right| \leq \frac{(1+\varepsilon)^3}{(1-\varepsilon)^3}\left(\frac{r_1}{r_2}\right)^K,
\end{aligned}$$

and therefore

$$\left(\frac{r_1}{r_2}\right)^K \geq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 \frac{\delta''}{\delta'}.$$

As a consequence, we obtain uniform bounds on the distortion of $\xi_n$ on the annulus $A$

$$\tilde{C}^{-1} \leq \left|\frac{\xi'_n(\lambda_1)}{\xi'_n(\lambda_2)}\right| \leq \tilde{C} \tag{22}$$

for all $\lambda_1, \lambda_2 \in A$, where $\tilde{C}$ depends only on $\delta''$ and $\delta'$.

In order to estimate the Lebesgue measure of the set $\{\lambda \in B(\lambda_0, r_2) : \xi_{n+m} \in U_{3\delta/4}\}$ for any radius $0 < r_2 \leq r$ and appropriate $m \leq N$, let us denote

$$E = \{z \in D : f^m_{\lambda_0}(z) \in U_{\delta/2}\}$$

and fix an arbitrary point $z_0 \in A$. By (21), we have $\xi_n^{-1}(E) \subset A$, and hence, by (22),

$$\mu(E) \leq \int_{\xi_n^{-1}(E)} |\xi_n'(z)|^2 d\mu(z) \leq \tilde{C}^2 |\xi_n'(z_0)|^2 \mu(\xi_n^{-1}(E)).$$

On the other hand, since the degree of $\xi_N$ is bounded by $K$ on $A$,

$$\mu(A) = \int_D \sum_{z \in \xi_n^{-1}(w) \cap A} |\xi_n'(z)|^{-2} d\mu(w) \leq \tilde{C}^2 K |\xi_n'(z_0)|^{-2} \mu(D).$$

Therefore, by (20) and since $r_1/r_2 \leq 0.1$ (see Lemma 15), we get the following inequalities

$$\begin{aligned} \mu(\xi_n^{-1}(E)) \geq & \tilde{C}^{-2} |\xi_n'(z_0)|^{-2} \mu(E) \geq \tilde{C}^{-2} |\xi_n'(z_0)|^{-2} C \mu(D) \\ \geq & \frac{C\tilde{C}^{-4}}{K} \mu(A) \geq \frac{C\tilde{C}^{-4}}{K} \frac{99}{100} \mu(B). \end{aligned}$$

Thus, for some $q \in (0,1)$, $q = q(\delta', \delta'', \delta)$, we have that

$$\mu\left(\xi_n^{-1}(E)\right) \geq q\mu(B).$$

By (21), this implies that

$$\mu\left(\{\lambda \in B : \xi_j(\lambda) \in U_{3\delta/4} \text{ for some } j \geq n\}\right) \geq q\mu(B).$$

By (19), if the critical value $e_\lambda$ falls under $f_\lambda$ to $U_{3\delta/4}$, then the parameter $\lambda$ cannot be in $\mathcal{M}_\delta$, so

$$\mu\left(\{\lambda \in B(\lambda_0, r_2) : \lambda \notin \mathcal{M}_\delta\}\right) \geq q\mu(B(\lambda_0, r_2)).$$

Since it holds for an arbitrarily small $r_2 \leq r$, the Lebesgue density of the set $\mathcal{M}_\delta$ at $\lambda_0$ is at most $1 - q < 1$. This completes the proof of Theorem 7.

# 4. PROOF OF LEMMA 4

To finish the proof of Theorem 2 we need to deal with the case when all critical values are prepoles. First, recall that every Weierstrass elliptic function has a countable family of poles which are exactly the lattice points. Poles of $f_\lambda$ are given by

$$p_{j,k}(\lambda) = j\lambda + k e^{2\pi i/3} \lambda, \quad j,k \in \mathbb{Z}$$

for $f_\lambda \in \mathcal{W}_t$ and by

$$p_{j,k}(\lambda) = j\lambda + ki\lambda, \quad j,k \in \mathbb{Z}$$

for $f \in \mathcal{W}_s$. These are obviously analytic functions of $\lambda$.

Suppose now that $\lambda_0$ is a parameter for which all critical values of $f_{\lambda_0} \in \mathcal{W}_t \cup \mathcal{W}_s$ are prepoles, i.e.

$$f^n_{\lambda_0}(e_{\lambda_0}) = p_{j,k}(\lambda_0) \tag{23}$$

for some $n \geq 0$. In case of a triangle lattice, $e_{\lambda_0}$ is any of the three critical values (then for remaining critical values we have analogous equations multiplied by $\mathrm{e}^{2\pi i/3}$ and $\mathrm{e}^{4\pi i/3}$, respectively), while for a square lattice we take $e_{\lambda_0} \neq 0$.

Consider the function

$$g(\lambda) = f^n_\lambda(e_\lambda) - p_{j,k}(\lambda)$$

in a neighbourhood of $\lambda_0$, where numbers $j,k \in \mathbb{Z}$ and $n \in \mathbb{N}$ are fixed. It is a holomorphic function of $\lambda$ for $\lambda$ close to $\lambda_0$, and by (23) we have $g(\lambda_0) = 0$. We have two cases: either $g$ is an open map and $\lambda_0$ is its isolated root, or $g(\lambda) \equiv 0$ locally.

If the second condition holds, for all parameters $\lambda$ close to $\lambda_0$, the dynamics of critical values is the same. To be precise, all critical values of $f_\lambda$ are mapped onto fixed poles after fixed number of iterates. We can argue exactly like in the proof of transversality (Lemma 9) – parameter $\lambda_0$ is postsingularly stable and we can find a conjugacy between $f_\lambda$ and $f_{\lambda_0}$ defined on branches of consecutive preimages of critical values. The conjugacy may be extended to a quasiconformal map on the Julia set $J(f_{\lambda_0})$, conjugating $f_{\lambda_0}$ with $f_\lambda$ for all $\lambda$ close to $\lambda_0$. Therefore, there exists an $f_{\lambda_0}$-invariant line-field on $J(f_{\lambda_0})$, contrary to [10, Theorem 1.1] (cf. [4, Theorem 2]). This case cannot happen.

It implies that $g$ is not constant, and hence $\lambda_0$ is its isolated root. Consequently, there is no $\lambda$ close to $\lambda_0$ for which critical values of $f_\lambda$ are eventually mapped onto these poles after $n$ iterates (in the case of a square lattice, this does not concern 0, which is always a pole), hence the set of parameters satisfying (23) is discrete. Since there are only countably many such equations, we conclude that the set of parameters $\lambda$ for which all critical values of $f_\lambda$ are prepoles is countable. This completes the proof of Lemma 4.

Notice that this does not prove that the whole set of parameters for which all critical values are prepoles is discrete. Moreover, results of Jane Hawkins and her collaborates show that these parameters accumulate similarly to a family of consecutive prepoles of a meromorphic function. Still, they form a countable set with the Lebesgue measure zero in $\mathbb{C}$.

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